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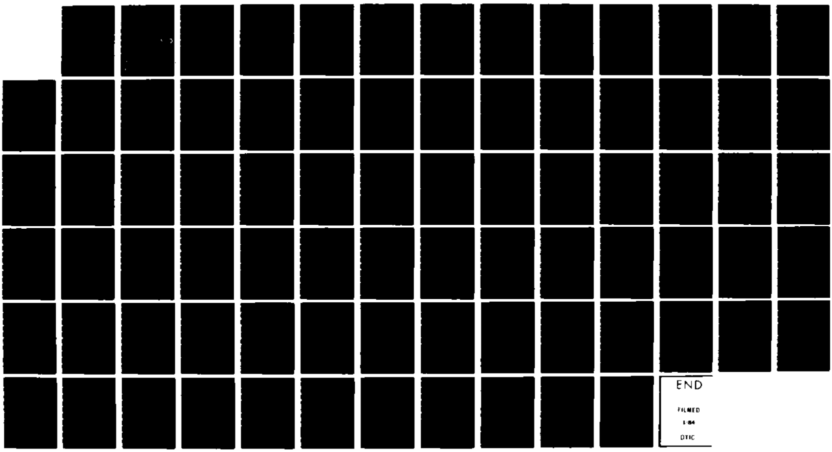

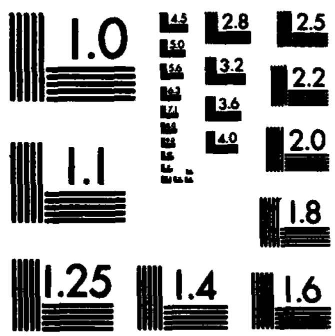

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ABSTRACT

JENSEN, DAVID WARREN. Derivations of a Prime Ring Which Satisfy a Polynomial Identity. (Under the direction of JIANG LUH).

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Let λ_j, j = 1,2,3,4, and 5 be derivations of a prime ring R, and let Q(R) be the generalized ring of quotients of R. Assume m and n are positive integers, C is the center of Q(R), and Λ_j is the unique derivation of Q(R) satisfying $\Lambda_j|_R = \lambda_j$. lambda sub j. Several identities involving lambda sub j the λ_j's are studied and the following results are established.

1. If $\lambda_1^n = \lambda_2$, $n > 1$, then

   a) $\lambda_1 = 0$ if R is commutative and char R is sufficiently large.

   b) λ_1 is algebraic if n = 3,4,5, or 6, and char R is sufficiently large.

   c) $\lambda_1^{2n-3} = 0$ if $\exists$ $a \in R$ such that $\lambda_1 a \neq 0$ and $\lambda_1^2 a = 0$.

   d) λ_1 is algebraic if $\Lambda_1(C) = 0$, char R = 0, and if $\exists$ $0 \neq a \in R$ and $0 \neq c \in C$ such that $\lambda_1 a = ca$.

2. If $\lambda_1\lambda_2^m = 0$, then either $\lambda_1 = 0$ or $\lambda_2^k = 0$ where $k \leq 4m-1$.

3. If $\lambda_1^n\lambda_2 = 0$, then either $\lambda_1^k = 0$ or $\lambda_2^2 = 0$ where $k \leq 12n-9$.

4. If $\lambda_1^n\lambda_2^m = 0$ and $\lambda_1\lambda_2 = \lambda_2\lambda_1$, then either λ_1 or λ_2 is nilpotent.

5. If $\lambda_1\lambda_2 - \lambda_3\lambda_4 = \lambda_5$, $\lambda_j \neq 0$ for j = 1,2,3, and 4, and char R $\neq$ 2, then $\exists$ $c \in C$ such that

   either a) $\lambda_1 = c\lambda_3$ and $\lambda_2 = c^{-1}\lambda_4$

   or b) $\lambda_1 = c\lambda_4$ and $\lambda_2 = c^{-1}\lambda_3$.

6. If $\lambda_1\lambda_2 - \lambda_3^3 = \lambda_4$ and char R $\neq$ 2, then either $\lambda_1 = 0$ or $\lambda_2 = 0$.

7. If $\lambda_1\lambda_2\lambda_1 = 0$ and neither λ_1 nor λ_2 is nilpotent, then λ_1^{2k+1} is a derivation, $\lambda_1^2\lambda_2^2 = \lambda_2^2\lambda_1^2$, and $\lambda_1\lambda_2^{2k+1}\lambda_1 = 0$, $\forall\, k \in Z^+$.

a

8. If $\lambda_1\lambda_2^2\lambda_1 = 0$, char $R \neq 2$, and R has no zero divisors, then either λ_1 or λ_2 is nilpotent.

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Derivations of a Prime Ring Which Satisfy a Polynomial Identity

Capt, USAF

1983

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Doctor of Philosophy

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Let λ_j, $j = 1,2,3,4$, and 5 be derivations of a prime ring R, and let Q(R) be the generalized ring of quotients of R. Assume m and n are positive integers, C is the center of Q(R), and Λ_j is the unique derivation of Q(R) satisfying $\Lambda_j|_R = \lambda_j$. Several identities involving the λ_j's are studied and the following results are established:

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8. If $\lambda_1\lambda_2^2\lambda_1 = 0$, char $R \neq 2$, and R has no zero divisors, then either λ_1 or λ_2 is nilpotent.

DERIVATIONS OF A PRIME RING WHICH SATISFY A POLYNOMIAL IDENTITY

by

DAVID WARREN JENSEN

A thesis submitted to the Graduate Faculty of
North Carolina State University at Raleigh
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

DEPARTMENT OF MATHEMATICS

RALEIGH

1983

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BIOGRAPHY

David Warren Jensen is a Distinguished Graduate of the United States Air Force Academy, where he received his Bachelor of Science Degree and his Commission as an Air Force Officer in 1973. Immediately after graduating from the Academy, he attended Colorado State University, obtaining a Master of Science Degree in Applied Mathematics in 1974. Also in 1974, he completed Undergraduate Navigator Training at Mather Air Force Base, California. From California, he moved to McGuire Air Force Base, New Jersey, where he served for three years as an Instructor Navigator in the C-141 aircraft.

In 1978, Capt. Jensen returned to the Air Force Academy and assumed the position of Assistant Professor in the Department of Mathematical Sciences. He was sponsored by the Air Force in 1980 to begin full-time graduate study in Mathematics at North Carolina State University.

Capt. Jensen, who was born in Munich, Germany, on November 28, 1951, grew up in Colorado Springs, Colorado. He is married to the former Nancy Jane Congdon from Pueblo, Colorado, and he has a five-month-old daughter, Laura.

ACKNOWLEDGEMENTS

For his guidance, patience, and encouragement, I would like to thank my advisor, Dr. Jiang Luh. I also wish to express my appreciation to Dr. Lung O. Chung for his many creative ideas and friendship. Most importantly, I thank the Lord for His sustaining grace during the past three years.

TABLE OF CONTENTS

1. INTRODUCTION

During the past thirty years there have been several milestones in the study of algebraic derivations in prime rings. In 1957, Amitsur [1] proved a famous theorem: in a simple ring with unity, any derivation λ which satisfies a polynomial identity $f(\lambda) = 0$ must be inner. In 1978, Kharchenko [21] generalized the above result to prime rings: a derivation satisfying a polynomial identity in a prime ring R with characteristic zero may be extended to an inner derivation of the generalized ring of quotients of R. Chung, Kovacs, and Luh [4] have recently sharpened Kharchenko's result and answered several major questions, including what type of minimal polynomial a derivation can have.

The major results by Amitsur, Kharchenko, Chung, Kovacs, and Luh tell us a great deal when a derivation λ satisfies a polynomial identity $f(\lambda) = 0$. However, relatively little is known when more than one derivation is involved, i.e., what can be said when derivations $\lambda_1, \lambda_2, \ldots, \lambda_n$ of a prime ring R satisfy a polynomial identity of the form $f(\lambda_1, \lambda_2, \ldots, \lambda_n) = 0$? A general theory for this problem appears to be beyond our current reach. The purpose of this report is to establish a base of knowledge by investigating several specific polynomials.

A natural starting point for the study of polynomial identities of the form $f(\lambda_1, \lambda_2) = 0$ is a well-known result by Posner [26]: if λ, δ, and γ are derivations of a 2-torsion free, prime ring R such that $\lambda\delta = \gamma$, then either $\lambda = 0$ or $\delta = 0$. It follows trivially that

$f(\lambda,\delta) = \lambda^2 - \delta = 0$ implies $\lambda = 0$, and that $f(\lambda,\delta) = \lambda\delta = 0$ implies either $\lambda = 0$ or $\delta = 0$. From these observations two logical questions arise: what can be said when $f(\lambda,\delta) = \lambda^n - \delta = 0$ for $n > 1$ (i.e., the case where λ and an iterate λ^n are both derivations), and what can be said when $f(\lambda,\delta) = \lambda^n\delta^m = 0$ for $n,m \in Z^+$? These two cases are studied and several results given in Chapters 3 and 4. In Chapter 5 identities involving more than two variables are considered. Here, among other things, we extend Posner's result by determining what happens when $\lambda\delta = \gamma^2$ and $\lambda\delta = \gamma^3$. In Chapter 6 we investigate the identities $\lambda\delta\lambda = 0$ and $\lambda\delta^2\lambda = 0$, and in Chapter 7 we make some concluding remarks. Chapter 2 introduces the generalized ring of quotients $Q(R)$ of a prime ring R, the unique derivation Λ of Q which satisfies $\Lambda|_R = \lambda$, and other concepts which will be needed in the sequel. Throughout this paper Z is the ring of integers, Z^+ is the set of positive integers, and $m,n \in Z^+$. Also G, C, and F denote the center of R, the center of $Q(R)$, and the algebraic closure of C, respectively.

The major results of Chapters 3, 4, 5, and 6 are summarized below.

Chapter 3. Assuming R is a prime ring and λ and λ^n, $n > 1$, are derivations of R:

1. If R is commutative and characteristic R is sufficiently large, then $\lambda = 0$.
2. If $n = 3$ and characteristic $R \neq 3$, then λ is algebraic and $\lambda^3 = c\lambda$ for some $c \in C$.
3. If $n = 4$ and characteristic $R \neq 2$, then λ is algebraic and $\{\lambda,\lambda^3,\lambda^4,\lambda^6\}$ is linearly dependent over C.
4. If $n = 5$ and characteristic $R \neq 5$, then λ is algebraic and $\lambda^7 = c\lambda$ for some $c \in C$.

5. If $n = 6$ and characteristic R is sufficiently large, then λ is algebraic.

6. If $\exists\ a \in R$ such that $\lambda a \neq 0$ and $\lambda^2 a = 0$, then $\lambda^{2n-3} = 0$. If, in addition, characteristic $R \neq 2$, then $\lambda^n = 0$ if n is odd and $\lambda^{n-1} = 0$ if n is even.

7. If $\Lambda(C) = 0$, characteristic $R = 0$, and $\exists\ 0 \neq a \in R$ and $0 \neq c \in C$, such that $\lambda a = ca$, then λ is algebraic.

Chapter 4. Assuming R is a prime ring and λ and δ are derivations of R:

1. If $\lambda\delta^m = 0$, then either $\lambda = 0$ or $\delta^k = 0$, $k \leq 4m - 1$.
2. If $\lambda^n\delta = 0$, then either $\lambda^k = 0$ or $\delta^2 = 0$, $k \leq 12n - 9$.
3. If $\lambda^n\delta^m = 0$ and $\lambda\delta = \delta\lambda$, then either λ is nilpotent or δ is nilpotent.

Chapter 5. Assuming R is a prime ring, characteristic $R \neq 2$, and $\lambda, \delta, \gamma, \varepsilon$, and σ are derivations of R:

1. If λ, δ, γ, and ε are nonzero and $\lambda\delta - \gamma\varepsilon = \sigma$, then $\exists\ c \in C$ such that either a) $\lambda = c\gamma$ and $\delta = c^{-1}\varepsilon$
   or b) $\lambda = c\varepsilon$ and $\delta = c^{-1}\gamma$.
2. If $\delta\gamma - \lambda^3 = \sigma$, then either $\delta = 0$ or $\gamma = 0$.

Chapter 6. Assuming R is a prime ring and λ and δ are derivations of R:

1. If $\lambda\delta\lambda = 0$ and neither λ nor δ is nilpotent, then λ^{2k+1} is a derivation, $\lambda^2\delta^2 = \delta^2\lambda^2$, and $\lambda\delta^{2k+1}\lambda = 0$, $\forall\ k \in Z^+$.
2. If $\lambda\delta^2\lambda = 0$, characteristic $R \neq 2$, and R has no zero divisors, then either λ is nilpotent or δ is nilpotent.

2. PRELIMINARIES

2.1 Definitions

A prime ring R is a ring with the property that for $a,b \in R$, if $axb = 0$ for all $x \in R$, then either $a = 0$ or $b = 0$. Primitive rings, integral domains, and simple rings with $R^2 \neq 0$, are all examples of prime rings. A ring R is called a semi-prime ring if for $a \in R$, $axa = 0$ for all $x \in R$ implies $a = 0$. It follows that every prime ring must be semi-prime. However, the converse need not be true. (For example, $Z \oplus Z$, the direct sum of two copies of Z, is semi-prime but not prime.) A derivation of a ring R is an additive mapping $\lambda: R \to R$ satisfying $\lambda(xy) = \lambda xy + x\lambda y$ for all $x,y \in R$. A derivation λ is called an inner derivation if there exists an element a in R such that $\lambda x = ax - xa$, for all x in R.

If R is a nonzero ring and there exists a positive integer n such that $na = 0$, $\forall\, a \in R$, we call the smallest such positive integer the characteristic of R. If no such positive integer exists, R is said to have characteristic zero. If R is a prime ring then the characteristic of R is either zero or a prime number. We say a ring R is n-torsion free if $nx = 0$ implies $x = 0$, $\forall\, x \in R$. If R is n-torsion free then characteristic $R \neq n$. If R is prime and characteristic $R > n$, then R is m-torsion free, $\forall\, m \leq n$.

2.2 The Generalized Ring of Quotients of a Prime Ring

In Chapter 3, we will make extensive use of the notion of the generalized ring of quotients of a prime ring R. Here we offer a brief development of this notion, similar to the developments presented in [4] and [16].

Given a prime ring R, let L denote the set of all nonzero, two-sided ideals of R. Let $\overline{Q(R)} = \{(U,f) \mid U \in L \text{ and } f \in \mathrm{Hom}_R(U_R,R_R)\}$, where $\mathrm{Hom}_R(U_R,R_R)$ is the set of all right R-homomorphisms from U_R into R_R. We define a relation $\sim$ on $\overline{Q(R)}$ by $(U,f) \sim (V,g)$ iff $f = g$ on a nonzero ideal $W \subseteq U \cap V$. Since R is prime it is trivial to show that $\sim$ is indeed an equivalence relation on $\overline{Q(R)}$. Denote by $[U,f]$ the equivalence class containing (U,f) and by $Q(R)$, or just Q, the set of equivalence classes. Addition and multiplication are then defined by $[U,f] + [V,g] = [U \cap V, f + g]$ and $[U,f][V,g] = [VU, fg]$, where the product fg is the composition of functions. With these operations it is a straightforward exercise to verify that Q is an associative ring. We call Q the generalized ring of quotients of R.

A very important property of Q which follows from the definition is the following:

If $q \in Q$, $q \neq 0$, then $\exists$ a nonzero ideal $U \subseteq R$ such that $qU \subseteq R$ and $qU \neq 0$. (2.1)

Using this property we can show that the generalized ring of quotients is itself a prime ring. Assume $q_1Qq_2 = 0$ where q_1 and q_2 are nonzero elements of Q. Since $q_1 \neq 0$ and $q_2 \neq 0$, $\exists$ nonzero ideals U_1 and U_2 such that $q_1U_1 \neq 0$ and $q_2U_2 \neq 0$. Let $U = U_1U_2$ and note that $U \neq 0$ because R is prime. Then $q_1U \neq 0$ and $q_2U \neq 0$ since q_1 and q_2 are nonzero. However, $q_1Qq_2 = 0$ implies $q_1(UR)q_2(U) = (q_1U)R(q_2U) = 0$ and by the primeness of R we get either $q_1U = 0$ or $q_2U = 0$. The contradiction tells us that $q_1Qq_2 = 0$ implies either $q_1 = 0$ or $q_2 = 0$ and we conclude that Q is a prime ring. It is also easy to show that

the characteristic of R is equal to the characteristic of Q, and that R is isomorphically embedded in Q via the map $a \to [R, a_\ell]$, where a_ℓ represents left multiplication by the element a.

If we let C represent the center of the generalized ring of quotients Q, then it is obvious that C has a unity since [R, 1] is a unity for the ring Q. Further we can prove that C is a field. Let $c \in C$, $c \neq 0$. Using (2.1) we know there exists a nonzero ideal $U \subseteq R$ such that $cU \neq 0$. Moreover cU is itself an ideal in R and we can define $h : cU \to R$ by $h(cu) = u$. Then h is a right R-homomorphism and by letting $d = [cU,h]$ we get $dc = cd = 1$ on U.

Besides being a field, C is especially nice in that it is precisely the set of elements in Q which commute with all of R. To prove this, start with $w \in Q$, $w \neq 0$, and $q \in Q$, $qx = xq$, $\forall\, x \in R$. Again by (2.1), $w \neq 0$ implies $\exists$ a nonzero ideal $U \subseteq R$ such that $wU \subseteq R$ and $wU \neq 0$. Then for all $u \in U$, $(qw)u = q(wu) = (wu)q = w(uq) = w(qu) = (wq)u$ and it follows that $qw = wq$. Therefore $q \in C$ and the proof is completed.

We have noted that R and Q are both prime rings with the same characteristic value and that R is isomorphically embedded in Q. In addition C is a field and $C = \{q \in Q \mid qx = xq, \forall\, x \in R\}$. With this close relationshiop between R and Q one might hope that derivations of R would extend nicely to all of Q. This is indeed the case and we have the following powerful result: If $\lambda : R \to R$ is a derivation, then there exists a unique extension $\Lambda : Q \to Q$ such that Λ is a derivation of Q and $\Lambda|_R = \lambda$. Given $[U,f] \in Q$, Λ is defined by $\Lambda([U,f]) = [U^2, f']$, where $f'(u) = \lambda(f(u)) - f(\lambda(u))$. Note that U^2 is used to insure $\lambda(u) \in U$ so that $f(\lambda(u))$ makes sense. Also note f' is an element of

$\mathrm{Hom}_R(U_R^2, R_R)$ since for any $r \in R$, $f'(ur) = \lambda(f(ur)) - f(\lambda(ur))$ $= \lambda(f(u))r + f(u)\lambda(r) - f(\lambda(u))r - f(u)\lambda(r) = f'(u)r$. Proving Λ is a derivation of Q is a routine exercise. To see $\Lambda|_R = \lambda$, let $x \in R$, and use the definition of Λ to get $\Lambda([R, x_\ell]) = [R^2, x_\ell']$, where $x_\ell'(r) = \lambda(xr) - x\lambda r = (\lambda x)r = (\lambda x)_\ell r$. Therefore $\Lambda([R, x_\ell]) = [R^2, (\lambda x)_\ell] = [R, (\lambda x)_\ell]$. To prove the uniqueness of Λ let $q \in Q$, $q \neq 0$, and assume Λ is not unique, say $\lambda = \Lambda_1|_R$ and $\lambda = \Lambda_2|_R$, where Λ_1 and Λ_2 are derivations of Q. By (2.1), $q \neq 0$ implies $\exists$ an ideal $U \subseteq R$, $U \neq 0$, such that $qU \subseteq R$ and $qU \neq 0$. Then $\Lambda_1(qu) = \Lambda_1 qu + q\Lambda_1 u$ $= \Lambda_2(qu) = \Lambda_2 qu + q\Lambda_2 u$ implies $(\Lambda_1 - \Lambda_2)qu = 0$, $\forall\, u \in U$. It follows that $(\Lambda_1 - \Lambda_2)q = 0$ and $\Lambda_1 q = \Lambda_2 q$.

A derivation $\lambda: R \to R$ satisfies a polynomial identity over C if $\exists$ a polynomial $p(t) = a_o + a_1 t + a_2 t^2 + \ldots + a_n t^n$, $a_i \in C$, $a_n \neq 0$, such that $p(\lambda)x = (a_o + a_1\lambda + a_2\lambda^2 + \ldots + a_n\lambda^n)x = 0$, $\forall\; x \in R$. *A derivation* which satisfies a polynomial identity over C is called algebraic over C.

As a final note, if the field C is not algebraically closed, let F be the algebraic closure of C. As discussed in [11] and [24], we can define $S = RC + C$, a closed prime algebra over C, and $P = S \otimes_C F$, a prime algebra over F. A derivation λ of R can be extended uniquely to P. We say λ is algebraic over F if it satisfies a polynomial identity with coefficients from F.

2.3 Propositions

We conclude this chapter with several propositions which will be needed later. Theorem 2.9 is similar to Proposition 2.8 and may be of some independent interest [3]. For easy reference, we repeat the results by Posner and Kharchenko mentioned in the Introduction.

Proposition 2.1

(Posner) [26]. If λ and δ are derivations of a prime ring R with characteristic R $\neq 2$, and if the composition $\lambda\delta$ is also a derivation of R, then either $\lambda = 0$ or $\delta = 0$.

Proposition 2.2

(Kharchenko) [21]. A derivation satisfying a polynomial identity in a prime ring R with characteristic zero can be uniquely extended to an inner derivation of the generalized ring of quotients of R.

Proposition 2.3

[4] . A derivation λ satisfying a polynomial identity in a prime ring R with characteristic zero must satisfy a minimal polynomial of the form $\psi(x) = x^{n_o} \prod_{i=1}^{t} (x - c_i)^{n_i} (x + c_i)^{n_i}$, where the c_i are mutually distinct elements of F; $n_o \geq n_i$, $\forall_i$; n_o is odd; and $n_j \geq \frac{n_o - 1}{2}$ for some $j \neq 0$.

Proposition 2.4

[16] . Suppose a_i, b_i are nonzero elements in a prime ring R such that $\sum_{i=1}^{t} a_i \, x \, b_i = 0$, for all $x \in R$. Then the a_i are linearly dependent over C and the b_i are linearly dependent over C.

Proposition 2.5

[7] . Let R be a 2-torsion free, semi-prime ring and λ be a derivation of R. If λ is nilpotent then the index of nilpotency is an odd number.

Proposition 2.6

[5] [12]. If R is a semi-prime ring and λ is a derivation of R such that $(\lambda x)^n = 0$ for fixed $n \in Z^+$ and for all $x \in R$, then $\lambda = 0$.

Proposition 2.7

[14]. Let R be a prime ring, characteristic $R \neq 2$, and suppose $a,b \in R$ are such that $axb + bxa = 0$, $\forall\, x \in R$. Then either $a = 0$ or $b = 0$.

Proposition 2.8

[4] [8]. Let R be a prime ring with characteristic zero and W be a nonzero ideal of R. Suppose λ is an algebraic derivation of R such that $\lambda W \subseteq W$ and $f(\lambda)\, W = (0)$. Then $f(\lambda)R = (0)$.

Theorem 2.9 Let R be a prime ring with characteristic zero and W be a noncentral Lie ideal of R. Suppose λ is a derivation of R such that $\lambda^n W = (0)$. Then $\lambda^h R = (0)$ for some $h \in Z^+$.

Proof: W is a Lie ideal iff W is an additive subgroup and

$$[u,x] = ux - xu \in W \qquad \forall\, u \in W,\ x \in R.$$

We know $[\lambda^n u,x] = 0$, $\forall\, u \in W,\ x \in R$. Let j be the least such that $[\lambda^j u,x] = 0$, $\forall\, u \in W,\ x \in R$. Clearly $j > 0$ since W is noncentral. Then

$$\lambda^n[u,x] = 0 \qquad \forall\, u \in W,\ x \in R$$

$$\Rightarrow \lambda^{n+1}[u,x] = \lambda^n(\lambda[u,x]) = \lambda^n[\lambda u,x] = 0 \qquad \forall\, u \in W,\ x \in R$$

$$\Rightarrow \lambda^{n+1}[\lambda u,x] = \lambda^n(\lambda[\lambda u,x]) = \lambda^n[\lambda^2 u,x] = 0 \quad \forall\, u \in W,\ x \in R$$

$$\Rightarrow \lambda^{n+1}[\lambda^2 u,x] = \lambda^n(\lambda[\lambda^2 u,x]) = \lambda^n[\lambda^3 u,x] = 0 \quad \forall\, u \in W,\ x \in R$$

$$\Rightarrow \ldots \Rightarrow \lambda^n[\lambda^{j-1} u,x] = 0 \qquad \forall\, u \in W,\ x \in R$$

$$[\lambda^{j-1} u,\lambda^n x] = 0 \qquad \forall\, u \in W,\ x \in R.$$

Let m be the least such that

$$[\lambda^{j-1} u,\lambda^m x] = 0 \qquad \forall\, u \in W,\ x \in R.$$

Notice that $m \geq 1$ since j was chosen smallest.

Replacing x by xy yields

$$[\lambda^{j-1}u, \lambda^m(xy)] = [\lambda^{j-1}u, \sum_{i=0}^{m} \binom{m}{i} \lambda^i x \lambda^{m-i} y] = 0 \quad \forall\, u \in W, x, y \in R.$$

Replacing x by $\lambda^m x$ and y by $\lambda^{m-1}y$ yields

$$[\lambda^{j-1}u, \lambda^{2m}x\lambda^{m-1}y] = 0 \qquad \forall\, u \in W, x, y \in R$$

$$\Rightarrow \quad \lambda^{2m}x[\lambda^{j-1}u, \lambda^{m-1}y] = 0 \qquad \forall\, u \in W, x, y \in R. \tag{2.2}$$

Replacing x by $(z\lambda^{2m-1}x)$ in (2.2) yields

$$\lambda^{2m}z\lambda^{2m-1}x[\lambda^{j-1}u, \lambda^{m-1}y] = 0 \qquad \forall\, u \in W, x, y, z \in R.$$

Replacing x by $(\lambda z\lambda^{2m-2}x)$ in (2.2) yields

$$\lambda^{2m+1}z\lambda^{2m-2}x[\lambda^{j-1}u, \lambda^{m-1}y] = 0 \qquad \forall\, u \in W, x, y, z \in R.$$

Replacing x by $(\lambda^2 z\lambda^{2m-3}x)$ in (2.2) yields

$$\lambda^{2m+2}z\lambda^{2m-3}x[\lambda^{j-1}u, \lambda^{m-1}y] = 0 \qquad \forall\, u \in W, x, y, z \in R.$$

$$\vdots$$

Eventually we obtain

$$\lambda^{4m-1}z\, x\, [\lambda^{j-1}u, \lambda^{m-1}y] = 0 \qquad \forall\, u \in W, x, y, z \in R.$$

By the primeness of R we may conclude $\lambda^{4m-1} = 0$ and the theorem is complete.

As encountered in the last proof, repeated use of the symbol $\forall$ becomes cumbersome when it is clear from the context that arbitrary elements are involved. Therefore in subsequent proofs, where no ambiguity exists, the repetitious use of $\forall$ will be omitted.

3. DERIVATIONS SATISFYING $f(\lambda,\delta) = \lambda^n - \delta = 0$

Consider the case where λ and an iterate λ^n are both derivations of a prime ring R. Proposition 2.1 tells us that if $n = 2$ and characteristic $R \neq 2$, then $\lambda = 0$. For $n \geq 2$, Martindale and Miers [24] have recently made several discoveries assuming λ is an inner derivation. In particular they have shown the following:

Proposition 3.1

If λ and λ^n are inner derivations of a prime ring R and characteristic R is sufficiently large, then λ is algebraic, and

1) if n is odd, then either $\lambda^n = 0$ or the minimal polynomial of λ is semisimple.
2) if n is even, then $\lambda^{n-1} = 0$.

We say a polynomial f is semisimple if f is the product of distinct irreducible linear factors over F. Note that if the minimal polynomial of λ is semisimple, Proposition 2.3 implies it must be of the form

$$\psi_\lambda(x) = x \prod_{i=1}^{t} (x - c_i)(x + c_i), \quad c_i \in F.$$

In this chapter we study the case where λ and λ^n are both derivations of a prime ring R, without the restriction that λ must be inner. We begin by assuming R is commutative.

3.1 Commutative Rings

Lemma 3.2 Let λ be a derivation of a commutative ring R, let $m,n \in Z^+$, $m < n$, and let N denote the natural numbers. If $\exists$ a function c from $\displaystyle\mathop{X}_{i=1}^{m} N$ to Z^+ such that $\forall\, x_1, x_2, \ldots, x_n \in R$,

$$\sum_{(k_1 + k_2 + \ldots + k_m = n - m)} c(k_1,k_2,\ldots,k_m)\lambda^{k_1+1} x_1 \lambda^{k_2+1} x_2 \ldots \lambda^{k_{m-1}+1} x_{m-1} \lambda^{k_m+1} x_m = 0$$

then $\exists$ a function d from $\overset{m+1}{\underset{i=1}{X}} N$ to Z^+ such that $\forall\, x_1,x_2,\ldots,x_{m+1} \in R$,

$$\sum_{(k_1+ k_2+\ldots+ k_{m+1}=n - m - 1)} d(k_1,k_2,\ldots,k_{m+1})\lambda^{k_1+1} x_1 \lambda^{k_2+1} x_2 \ldots \lambda^{k_m+1} x_m \lambda^{k_{m+1}+1} x_{m+1} = 0.$$

<u>Proof.</u> Using commutativity, label the k_i so that $k_m \neq 0$. Replacing x_m by $x_m x_{m+1}$ implies $\forall\, x_1,x_2,\ldots,x_m,x_{m+1} \in R$,

$$\sum_{(k_1 + k_2 + \ldots + k_m = n - m)} c(k_1,k_2,\ldots,k_m)\, \lambda^{k_1+1} x_1 \lambda^{k_2+1} x_2 \ldots \lambda^{k_{m-1}+1} x_{m-1} \lambda^{k_m+1} (x_m x_{m+1}) = 0$$

$$\Rightarrow \sum_{(k_1 + k_2 + \ldots + k_m = n - m)} c(k_1,k_2,\ldots,k_m)\, \lambda^{k_1+1} x_1\, \lambda^{k_2+1} x_2 \ldots \lambda^{k_{m-1}+1} x_{m-1}$$

$$\left(\sum_{j=0}^{k_m+1} \binom{k_m+1}{j}\, \lambda^{k_m+1-j} x_m \lambda^j x_{m+1} \right) = 0$$

$$\Rightarrow \sum_{(k_1 + k_2 + \ldots + k_m = n - m)} c(k_1,k_2,\ldots,k_m)\, \lambda^{k_1+1} x_1 \lambda^{k_2+1} x_2 \ldots \lambda^{k_{m-1}+1} x_{m-1}$$

$$\left(\sum_{j=1}^{k_m}\binom{k_m+1}{j}\lambda^{k_m+1-j}x_m\lambda^j x_{m+1}\right) = 0$$

$$\Rightarrow \sum_{(k_1+k_2+\ldots+k_m=n-m)} c(k_1,k_2,\ldots,k_m)\lambda^{k_1+1}x_1\lambda^{k_2+1}x_2\ldots\lambda^{k_{m-1}+1}x_{m-1}$$

$$\left(\sum_{j=0}^{k_m-1}\binom{k_m+1}{j+1}\lambda^{k_m-j}x_m\lambda^{j+1}x_{m+1}\right) = 0.$$

Letting $h = k_m - 1 - j$ yields

$$\sum_{(k_1+k_2+\ldots+k_{m-1}+h+j=n-m-1)} c(k_1,k_2,\ldots,k_{m-1},h+j+1)\lambda^{k_1+1}x_1\lambda^{k_2+1}x_2\ldots\lambda^{k_{m-1}+1}x_{m-1}$$

$$\left(\sum_{(h+j=k_m-1)}\binom{h+j+2}{j+1}\lambda^{h+1}x_m\lambda^{j+1}x_{m+1}\right) = 0.$$

$$\Rightarrow \sum_{(k_1+k_2+\ldots+k_{m-1}+h+j=n-m-1)} d(k_1,k_2,\ldots,k_{m-1},h,j)\lambda^{k_1+1}x_1\lambda^{k_2+1}x_2\ldots\lambda^{k_{m-1}+1}x_{m-1}$$

$$(\lambda^{h+1}x_m\lambda^{j+1}x_{m+1}) = 0$$

where $d(k_1,k_2,\ldots,k_{m-1},h,j) = c(k_1,k_2,\ldots,k_{m-1},h+j+1)\binom{h+j+2}{j+1}$.

Theorem 3.3 If λ and λ^n, $n > 1$, are derivations of a commutative, semi-prime ring R and characteristic R is sufficiently large, then $\lambda = 0$.

Proof. λ^n is a derivation implies $\forall x, y \in R$,

$$\lambda^n(xy) = \sum_{i=0}^{n} \binom{n}{i} \lambda^{n-i}x\lambda^i y = \lambda^n xy + x\lambda^n y$$

$$\Rightarrow \sum_{i=1}^{n-1} \binom{n}{i} \lambda^{n-i}x\lambda^i y = \sum_{i=0}^{n-2} \binom{n}{i+1} \lambda^{n-i-1}x\lambda^{i+1} y = 0$$

$$\Rightarrow \sum_{j+i=n-2} \binom{j+i+2}{i+1} \lambda^{j+1}x\lambda^{i+1} y = 0, \quad \text{where } j = n-2-i.$$

Applying the last lemma $n-2$ times we get $M\lambda x_1 \lambda x_2 \ldots \lambda x_n = 0$, $\forall x_1, x_2, \ldots, x_n \in R$, where $M \in Z^+$. If we let $x = x_i$ for $i = 1,2,\ldots,n$, then $M(\lambda x)^n = 0$, $\forall x \in R$. For characteristic $R > M$ we have $(\lambda x)^n = 0$, $\forall x \in R$. The proof is completed by using Proposition 2.6.

Corollary 3.4 If λ and λ^n, $n > 1$, are derivations of a semi-prime ring R with center G and characteristic R is sufficiently large, then $\lambda(G) = 0$.

Proof. It is trivial to show that G is also a prime ring and that any derivation of R is also a derivation of G. Therefore λ and λ^n are both derivations of G and by Theorem 3.3 we may conclude that $\lambda(G) = 0$.

3.2 General Rings

We now drop our assumptions that R is commutative and prime, and present three minor but interesting results. We will see Lemma 3.5 again in Chapter 4. One should compare Lemmas 3.6 and 3.7 to Theorems 3.9 and 3.11, where the primeness of R is required.

Lemma 3.5 If λ is a derivation of a ring R and $\lambda(\lambda x\lambda y) = 0$ $\forall x,y \in R$, then λ^{2n+1} is a derivation of R, $\forall$ $n \in Z^+$.

Proof. Observe that $\lambda(\lambda x\lambda y) = 0$, $\forall$ $x,y \in R$, implies $\lambda^h(\lambda^i x\lambda^j y) = 0$, $\forall$ $x,y \in R$ and $\forall$ $h,i,j \in Z^+$. We proceed by induction on n. Since $\lambda^3(xy) = \lambda^3xy + 3\lambda^2x\lambda y + 3\lambda x\lambda^2y + x\lambda^3y = \lambda^3xy + 3\lambda(\lambda x\lambda y) + x\lambda^3y$

$= \lambda^3xy + x\lambda^3y$, λ^3 is a derivation. Now assume λ^{2n+1} is a derivation for $n = 1,2,\ldots,k-1$. Then

$$\lambda^{2k+1}(xy) = \sum_{i=0}^{2k+1} \binom{2k+1}{i} \lambda^{2k+1-i}x\lambda^i y$$

$$= \lambda^{2k+1}xy + x\lambda^{2k+1}y + \sum_{i=1}^{2k} \binom{2k+1}{i} \lambda^{2k+1-i}x\lambda^i y$$

$$= \lambda^{2k+1}xy + x\lambda^{2k+1}y + \sum_{i=1}^{k} \binom{2k+1}{i} \lambda^{2k+1-2i}(\lambda^i x\lambda^i y)$$

$$= \lambda^{2k+1}xy + x\lambda^{2k+1}y.$$

Lemma 3.6 If λ and λ^3 are derivations of a ring R, characteristic $R \neq 3$, then λ^{2n+1} is a derivation of R, $\forall$ $n \in Z^+$.

Proof. If λ^3 is a derivation, then $\lambda^3(xy) = \lambda^3xy + 3\lambda^2x\,y +$ $3\lambda x\lambda^2y + x\lambda^3y = \lambda^3xy + x\lambda^3y$ and we get $3\lambda^2x\,y + 3\lambda x\lambda^2y = 3\lambda(\lambda x\lambda y) = 0$,

$\forall\, x,y \in R$. Since characteristic $R \neq 3$, $\lambda(\lambda x\lambda y) = 0$ and the proof is concluded by our last lemma.

<u>Lemma 3.7</u> If λ and λ^5 are derivations of a ring R, characteristic $R \neq 5$, then λ^7 is a derivation of R.

<u>Proof.</u> λ^5 is a derivation implies $\forall\, x,y \in R$,

$$\lambda^5(xy) = \sum_{i=0}^{5} \binom{5}{i} \lambda^{5-i}x\lambda^i y = \lambda^5 xy + x\lambda^5 y$$

$$\Longleftrightarrow \quad 5\lambda^4 x\lambda y + 10\lambda^3 x\lambda^2 y + 10\lambda^2 x\lambda^3 y + 5\lambda x\lambda^4 y = 0$$

$$\Longleftrightarrow \quad 5\lambda^3(\lambda x\lambda y) - 5\lambda^3 x\lambda^2 y - 5\lambda^2 x\lambda^3 y = 0$$

$$\Longleftrightarrow \quad 5\lambda^3(\lambda x\lambda y) - 5\lambda(\lambda^2 x\lambda^2 y) = 0$$

$$\Longleftrightarrow \quad \lambda^3(\lambda x\lambda y) - \lambda(\lambda^2 x\lambda^2 y) = 0\,. \tag{3.1}$$

λ^7 is a derivation iff $\forall\, x,y \in R$,

$$\lambda^7(xy) = \sum_{i=0}^{7} \binom{7}{i} \lambda^{7-i}x\lambda^i y = \lambda^7 xy + x\lambda^7 y$$

$$\Longleftrightarrow 7\lambda^6 x\lambda y + 21\lambda^5 x\lambda^2 y + 35\lambda^4 x\lambda^3 y + 35\lambda^3 x\lambda^4 y + 21\lambda^2 x\lambda^5 y + 7\lambda x\lambda^6 y = 0$$

$$\Longleftrightarrow \quad 7\lambda^5(\lambda x\lambda y) - 14\lambda^5 x\lambda^2 y - 35\lambda^4 x\lambda^3 y - 35\lambda^3 x\lambda^4 y - 14\lambda^2 x\lambda^5 y = 0$$

$$\Longleftrightarrow \quad 7\lambda^5(\lambda x\lambda y) - 14\lambda^3(\lambda^2 x\lambda^2 y) + 7\lambda^4 x\lambda^3 y + 7\lambda^3 x\lambda^4 y = 0$$

$$\Longleftrightarrow \quad 7\lambda^5(\lambda x\lambda y) - 14\lambda^3(\lambda^2 x\lambda^2 y) + 7\lambda(\lambda^3 x\lambda^3 y) = 0. \tag{3.2}$$

Since λ^5 is a derivation, (3.1) implies

$$[7\lambda^5(\lambda x\lambda y) - 7\lambda^3(\lambda^2 x\lambda^2 y)] + [-7\lambda^3(\lambda^2 x\lambda^2 y) + 7\lambda\,(\lambda^3 x\lambda^3 y)] = 0 + 0 = 0.$$

Therefore (3.2) is satisfied and λ^7 is a derivation.

3.3 Results for n = 3, 4, 5, and 6.

Returning to prime rings, our next lemma is a powerful variation of Proposition 2.4, giving us sufficient conditions for a derivation λ of a prime ring R to be algebraic. We will use it to obtain results for $f(\lambda,\delta) = \lambda^n - \delta = 0$, $n = 3,4,5$ and 6.

Lemma 3.8 Let $f_1, f_2, \ldots, f_m$ be functions of R into R, where R is prime, and let $y_1, y_2, \ldots, y_m$ be nonzero elements of R such that $f_1(x)zy_1 + f_2(x)zy_2 + \ldots + f_m(x)zy_m = 0$, $\forall\ x, z \in R$. Then $\exists$ $c_1, c_2, \ldots, c_m \in C$, not all zero, such that $c_1f_1 + c_2f_2 + \ldots + c_mf_m = 0$.

Proof. Note that for $m = 1$, $f_1(x)zy_1 = 0$, $\forall\ x, z \in R$, and $y_1 \neq 0$ implies $f_1 = 0$. Suppose the lemma is false and choose the least m. We just noted that $m \geq 2$ and we proceed by defining $c_k : Ry_mR \to Ry_kR$, $k = 1,2,\ldots,m$, by $c_k(\sum_i u_iy_mv_i) = \sum_i u_iy_kv_i$. We first show c_k is well defined. Assuming $\sum_i u_iy_mv_i = 0$ implies $\sum_i f_m(x)zu_iy_mv_i = 0$. Letting $z = zu_i$ in our initial hypothesis yields $f_1(x)zu_iy_1 + \ldots + f_m(x)zu_iy_m = 0$.

$$\text{Therefore}\quad f_m(x)zu_iy_m = -\sum_{j=1}^{m-1} f_j(x)zu_iy_j$$

$$\Rightarrow \sum_i \sum_{j=1}^{m-1} f_j(x)zu_iy_jv_i = 0$$

$$\Rightarrow \sum_{j=1}^{m-1} f_j(x)z\Big(\sum_i u_iy_jv_i\Big) = 0.$$

Since m was chosen to be least we must have $\sum_i u_iy_jv_i = 0$, $j = 1,2,\ldots,m-1$.

Therefore c_k is well defined. Also it is clear that c_k is a right

R-homomorphism and $[Ry_mR, c_k] \in Q(R)$. To see that $c_k \in C$, let $[U,g] \in Q(R)$ and consider $c_k g$ acting on URy_mR,

$c_k g(ur_1 y_m r_2) = c_k[g(ur_1)y_m r_2] = g(ur_1)y_k r_2 = g(ur_1 y_k r_2) = gc_k(ur_1 y_m r_2)$.

By hypothesis we have $\sum_{j=1}^{m} f_j(x)zy_j = 0$. Picking $r \in R$ such that $y_m r \neq 0$, we get $\sum_{j=1}^{m} f_j(x)zy_j r = 0$. Replacing $zy_j r$ by $c_j(zy_m r)$ yields $\sum_{j=1}^{m} f_j(x)c_j zy_m r = 0$. By the primeness of R, $\sum_{j=1}^{m} f_j(x)c_j = 0$.

<u>Theorem 3.9</u> If λ and λ^3 are derivations of a prime ring R, characteristic R $\neq 3$, then λ is algebraic and $\lambda^3 = c\lambda$ for some $c \in C$.

<u>Proof.</u> $\lambda^3(xy) = \lambda^3 xy + 3\lambda^2 x\lambda y + 3\lambda x\lambda^2 y + x\lambda^3 y = \lambda^3 xy + x\lambda^3 y$, $\forall x,y \in R$

$$\Rightarrow \quad 3\lambda^2 x\lambda y + 3\lambda x\lambda^2 y = 0$$

$$\Rightarrow \quad \lambda^2 x\lambda y + \lambda x\lambda^2 y = 0. \tag{3.3}$$

Replacing x by λx in (3.3) yields $\lambda^3 x\lambda y + \lambda^2 x\lambda^2 y = 0$.

Replacing x by xz yields $\lambda^3 xz\lambda y + x\lambda^3 z\lambda y + \lambda^2 xz\lambda^2 y + 2\lambda x\lambda z\lambda^2 y + x\lambda^2 z\lambda^2 y = 0$

$$\Rightarrow \quad \lambda^3 xz\lambda y + \lambda^2 xz\lambda^2 y + 2\lambda x\lambda z\lambda^2 y = 0$$

$$\Rightarrow \quad \lambda^3 xz\lambda y + \lambda^2 xz\lambda^2 y - 2\lambda x\lambda^2 z\lambda y = 0. \tag{3.4}$$

Replacing y by λy in (3.3) yields $\lambda^2 x\lambda^2 y + \lambda x\lambda^3 y = 0$.

Replacing y by zy yields $\lambda^2 x\lambda^2 zy + 2\lambda^2 x\lambda z\lambda y + \lambda^2 xz\lambda^2 y + \lambda x\lambda^3 zy + \lambda xz\lambda^3 y = 0$

$$\Rightarrow \quad \lambda xz\lambda^3 y + \lambda^2 xz\lambda^2 y + 2\lambda^2 x\lambda z\lambda y = 0$$

$$\Rightarrow \quad \lambda xz\lambda^3 y + \lambda^2 xz\lambda^2 y - 2\lambda x\lambda^2 z\lambda y = 0. \tag{3.5}$$

Subtracting (3.4) from (3.5) gives $\lambda xz\lambda^3 y - \lambda^3 xz\lambda y = 0$. This is the desired form for Lemma 3.8 and the proof is complete.

<u>Theorem 3.10</u> If λ and λ^4 are derivations of a prime ring R, characteristic $R \neq 2$, then λ is algebraic and $\{\lambda, \lambda^3, \lambda^4, \lambda^6\}$ is linearly dependent over C.

<u>Proof.</u> $\lambda^4(xy) = \lambda^4 xy + 4\lambda^3 x\lambda y + 6\lambda^2 x\lambda^2 y + 4\lambda x\lambda^3 y + x\lambda^4 y = \lambda^4 xy + x\lambda^4 y$, $\forall\, x, y \in R$

$$\Rightarrow \quad 4\lambda^3 x\lambda y + 6\lambda^2 x\lambda^2 y + 4\lambda x\lambda^3 y = 0$$

$$\Rightarrow \quad 2\lambda^3 x\lambda y + 3\lambda^2 x\lambda^2 y + 2\lambda x\lambda^3 y = 0. \tag{3.6}$$

Replacing x by λx in (3.6) yields $2\lambda^4 x\lambda y + 3\lambda^3 x\lambda^2 y + 2\lambda^2 x\lambda^3 y = 0$.

Replacing x by xz yields $2\lambda^4 xz\lambda y + 2x\lambda^4 z\lambda y + 3\lambda^3 xz\lambda^2 y + 9\lambda^2 x\lambda z\lambda^2 y + 9\lambda x\lambda^2 z\lambda^2 y + 3x\lambda^3 z\lambda^2 y + 2\lambda^2 xz\lambda^3 y + 4\lambda x\lambda z\lambda^3 y + 2x\lambda^2 z\lambda^3 y = 0$

$$\Rightarrow 2\lambda^4 xz\lambda y + 3\lambda^3 xz\lambda^2 y + 2\lambda^2 xz\lambda^3 y + 9\lambda x\lambda^2 z\lambda^2 y + 9\lambda^2 x\lambda z\lambda^2 y + 4\lambda x\lambda z\lambda^3 y = 0 \tag{3.7}$$

$$\Rightarrow 2\lambda^4 xz\lambda y + 3\lambda^3 xz\lambda^2 y + 2\lambda^2 xz\lambda^3 y + 9\lambda^2 x\lambda z\lambda^2 y - 6\lambda x\lambda^3 z\lambda y - 6\lambda x\lambda z\lambda^3 y + 4\lambda x\lambda z\lambda^3 y = 0$$

$$\Rightarrow 2\lambda^4 xz\lambda y + 3\lambda^3 xz\lambda^2 y + 2\lambda^2 xz\lambda^3 y + 9\lambda^2 x\lambda z\lambda^2 y - 6\lambda x\lambda^3 z\lambda y - 2\lambda x\lambda z\lambda^3 y = 0. \tag{3.8}$$

Replacing y by λy in (3.6) yields $2\lambda^3 x\lambda^2 y + 3\lambda^2 x\lambda^3 y + 2\lambda x\lambda^4 y = 0$.

Replacing y by zy yields $2\lambda^3 x\lambda^2 zy + 4\lambda^3 x\lambda z\lambda y + 2\lambda^3 xz\lambda^2 y + 3\lambda^2 x\lambda^3 zy + 9\lambda^2 x\lambda^2 z\lambda y + 9\lambda^2 x\lambda z\lambda^2 y + 3\lambda^2 xz\lambda^3 y + 2\lambda x\lambda^4 zy + 2\lambda xz\lambda^4 y = 0$

$$\Rightarrow 2\lambda xz\lambda^4y + 2\lambda^3xz\lambda^2y + 3\lambda^2xz\lambda^3y + 9\lambda^2x\lambda z\lambda^2y + 9\lambda^2x\lambda^2z\lambda y + 4\lambda^3x\lambda z\lambda y = 0 \quad (3.9)$$

$$\Rightarrow 2\lambda xz\lambda^4y + 2\lambda^3xz\lambda^2y + 3\lambda^2xz\lambda^3y + 9\lambda^2x\lambda z\lambda^2y - 6\lambda x\lambda^3z\lambda y - 6\lambda^3x\lambda z\lambda y + 4\lambda^3x\lambda z\lambda y = 0$$

$$\Rightarrow 2\lambda xz\lambda^4y + 2\lambda^3xz\lambda^2y + 3\lambda^2xz\lambda^3y + 9\lambda^2x\lambda z\lambda^2y - 6\lambda x\lambda^3z\lambda y - 2\lambda^3x\lambda z\lambda y = 0. \quad (3.10)$$

Subtracting (3.10) from (3.8) implies $2\lambda^4xz\lambda y - 2\lambda xz\lambda^4y + \lambda^3xz\lambda^2y - \lambda^2xz\lambda^3y - 2\lambda x\lambda z\lambda^3y + 2\lambda^3x\lambda z\lambda y = 0.$

Replacing x by λx and y by λy yields $2\lambda^5xz\lambda^2y - 2\lambda^2xz\lambda^5y + \lambda^4xz\lambda^3y - \lambda^3xz\lambda^4y - 2\lambda^2x\lambda z\lambda^4y + 2\lambda^4x\lambda z\lambda^2y = 0$

$$\Rightarrow 10\lambda^5xz\lambda^2y - 10\lambda^2xz\lambda^5y + 5\lambda^4xz\lambda^3y - 5\lambda^3xz\lambda^4y - 10\lambda^2x\lambda z\lambda^4y + 10\lambda^4x\lambda z\lambda^2y = 0. \quad (3.11)$$

Replacing x by λx in (3.7) yields $2\lambda^5xz\lambda y + 3\lambda^4xz\lambda^2y + 2\lambda^3xz\lambda^3y + 9\lambda^3x\lambda z\lambda^2y + 9\lambda^2x\lambda^2z\lambda^2y + 4\lambda^2x\lambda z\lambda^3y = 0.$

Replacing y by λy in (3.9) yields $2\lambda xz\lambda^5y + 2\lambda^3xz\lambda^3y + 3\lambda^2xz\lambda^4y + 9\lambda^2x\lambda z\lambda^3y + 9\lambda^2x\lambda^2z\lambda^2y + 4\lambda^3x\lambda z\lambda^2y = 0.$

Subtracting the last equation from the one preceeding it we get

$$2\lambda^5xz\lambda y - 2\lambda xz\lambda^5y + 3\lambda^4xz\lambda^2y - 3\lambda^2xz\lambda^4y + 5\lambda^3x\lambda z\lambda^2y - 5\lambda^2x\lambda z\lambda^3y = 0.$$

Taking λ of both sides implies $2\lambda^6xz\lambda y + 2\lambda^5xz\lambda^2y - 2\lambda^2xz\lambda^5y - 2\lambda xz\lambda^6y + 3\lambda^5xz\lambda^2y + 3\lambda^4xz\lambda^3y - 3\lambda^3xz\lambda^4y - 3\lambda^2xz\lambda^5y + 5\lambda^4x\lambda z\lambda^2y + 5\lambda^3x\lambda z\lambda^3y - 5\lambda^3x\lambda z\lambda^3y - 5\lambda^2x\lambda z\lambda^4y = 0$

$$\Rightarrow 2\lambda^6 xz\lambda y + 5\lambda^5 xz\lambda^2 y + 3\lambda^4 xz\lambda^3 y - 3\lambda^3 xz\lambda^4 y - 5\lambda^2 xz\lambda^5 y - 2\lambda xz\lambda^6 y$$

$$+ 5\lambda^4 x\lambda z\lambda^2 y - 5\lambda^2 x\lambda z\lambda^4 y = 0$$

$$\Rightarrow 4\lambda^6 xz\lambda y + 10\lambda^5 xz\lambda^2 y + 6\lambda^4 xz\lambda^3 y - 6\lambda^3 xz\lambda^4 y - 10\lambda^2 xz\lambda^5 y$$

$$- 4\lambda xz\lambda^6 y + 10\lambda^4 x\lambda z\lambda^2 y - 10\lambda^2 x\lambda z\lambda^4 y = 0. \qquad (3.12)$$

Subtracting (3.11) from (3.12) implies $4\lambda^6 xz\lambda y + \lambda^4 xz\lambda^3 y - \lambda^3 xz\lambda^4 y - 4\lambda xz\lambda^6 y = 0$. Now apply Lemma 3.8.

<u>Theorem 3.11</u> If λ and λ^5 are derivations of a prime ring R, characteristic R $\neq$ 5, then λ is algebraic and $\lambda^7 = c\lambda$ for some $c \in C$.

<u>Proof.</u> $\lambda^5(xy) = \sum_{i=0}^{5} \binom{5}{i} \lambda^{5-i} x\lambda^i y = \lambda^5 xy + x\lambda^5 y, \quad \forall\, x,y \in R$

$$\Rightarrow 5\lambda^4 x\lambda y + 10\lambda^3 x\lambda^2 y + 10\lambda^2 x\lambda^3 y + 5\lambda x\lambda^4 y = 0$$

$$\Rightarrow \lambda^4 x\lambda y + 2\lambda^3 x\lambda^2 y + 2\lambda^2 x\lambda^3 y + \lambda x\lambda^4 y = 0. \qquad (3.13)$$

Replacing x by λx in (3.13) yields $\lambda^5 x\lambda y + 2\lambda^4 x\lambda^2 y + 2\lambda^3 x\lambda^3 y + \lambda^2 x\lambda^4 y = 0$.

Replacing x by xz yields $\lambda^5 xz\lambda y + x\lambda^5 z\lambda y + 2\lambda^4 xz\lambda^2 y + 8\lambda^3 x\lambda z\lambda^2 y + 12\lambda^2 x\lambda^2 z\lambda^2 y + 8\lambda x\lambda^3 z\lambda^2 y + 2x\lambda^4 z\lambda^2 y + 2\lambda^3 xz\lambda^3 y + 6\lambda^2 x\lambda z\lambda^3 y + 6\lambda x\lambda^2 z\lambda^3 y + 2x\lambda^3 z\lambda^3 y + \lambda^2 xz\lambda^4 y + 2\lambda x\lambda z\lambda^4 y + x\lambda^2 z\lambda^4 y = 0$

$$\Rightarrow \lambda^5 xz\lambda y + 2\lambda^4 xz\lambda^2 y + 2\lambda^3 xz\lambda^3 y + \lambda^2 xz\lambda^4 y + 2\lambda x\lambda z\lambda^4 y$$

$$+ 6\lambda^2 x\lambda z\lambda^3 y + 8\lambda^3 x\lambda z\lambda^2 y + 6\lambda x\lambda^2 z\lambda^3 y + 12\lambda^2 x\lambda^2 z\lambda^2 y$$

$$+ 8\lambda x\lambda^3 x\lambda^2 y = 0. \qquad (3.14)$$

$$\Rightarrow \lambda^5 xz\lambda y + 2\lambda^4 xz\lambda^2 y + 2\lambda^3 xz\lambda^3 y + \lambda^2 xz\lambda^4 y + 2\lambda x\lambda z\lambda^4 y$$

$$+ 6\lambda^2 x\lambda z\lambda^3 y + 8\lambda^3 x\lambda z\lambda^2 y + 6\lambda x\lambda^2 z\lambda^3 y + 12\lambda^2 x\lambda^2 z\lambda^2 y$$

$$- 4\lambda x\lambda^4 z\lambda y - 8\lambda x\lambda^2 z\lambda^3 y - 4\lambda x\lambda z\lambda^4 y = 0$$

$$\Rightarrow \lambda^5 xz\lambda y + 2\lambda^4 xz\lambda^2 y + 2\lambda^3 xz\lambda^3 y + \lambda^2 xz\lambda^4 y - 2\lambda x\lambda z\lambda^4 y + 6\lambda^2 x\lambda z\lambda^3 y$$

$$+ 8\lambda^3 x\lambda z\lambda^2 y - 2\lambda x\lambda^2 z\lambda^3 y + 12\lambda^2 x\lambda^2 z\lambda^2 y - 4\lambda x\lambda^4 z\lambda y = 0. \quad (3.15)$$

Replacing y by λy in (3.13) yields $\lambda^4 x\lambda^2 y + 2\lambda^3 x\lambda^3 y + 2\lambda^2 x\lambda^4 y + \lambda x\lambda^5 y = 0$.

Replacing y by zy yields $\lambda^4 x\lambda^2 zy + 2\lambda^4 x\lambda z\lambda y + \lambda^4 xz\lambda^2 y + 2\lambda^3 x\lambda^3 zy$

$$+ 6\lambda^3 x\lambda^2 z\lambda y + 6\lambda^3 x\lambda z\lambda^2 y + 2\lambda^3 xz\lambda^3 y + 2\lambda^2 x\lambda^4 zy + 8\lambda^2 x\lambda^3 z\lambda y$$

$$+ 12\lambda^2 x\lambda^2 z\lambda^2 y + 8\lambda^2 x\lambda z\lambda^3 y + 2\lambda^2 xz\lambda^4 y + \lambda x\lambda^5 zy + \lambda xz\lambda^5 y = 0$$

$$\Rightarrow \lambda xz\lambda^5 y + \lambda^4 xz\lambda^2 y + 2\lambda^3 xz\lambda^3 y + 2\lambda^2 xz\lambda^4 y + 2\lambda^4 x\lambda z\lambda y + 6\lambda^3 x\lambda z\lambda^2 y$$

$$+ 8\lambda^2 x\lambda z\lambda^3 y + 6\lambda^3 x\lambda^2 z\lambda y + 12\lambda^2 x\lambda^2 z\lambda^2 y + 8\lambda^2 x\lambda^3 z\lambda y = 0 \quad (3.16)$$

$$\Rightarrow \lambda xz\lambda^5 y + \lambda^4 xz\lambda^2 y + 2\lambda^3 xz\lambda^3 y + 2\lambda^2 xz\lambda^4 y + 2\lambda^4 x\lambda z\lambda y + 6\lambda^3 x\lambda z\lambda^2 y$$

$$+ 8\lambda^2 x\lambda z\lambda^3 y + 6\lambda^3 x\lambda^2 z\lambda y + 12\lambda^2 x\lambda^2 z\lambda^2 y - 4\lambda x\lambda^4 z\lambda y - 8\lambda^3 x\lambda^2 z\lambda y$$

$$- 4\lambda^4 x\lambda z\lambda y = 0$$

$$\Rightarrow \lambda xz\lambda^5 y + \lambda^4 xz\lambda^2 y + 2\lambda^3 xz\lambda^3 y + 2\lambda^2 xz\lambda^4 y - 2\lambda^4 x\lambda z\lambda y + 6\lambda^3 x\lambda z\lambda^2 y$$

$$+ 8\lambda^2 x\lambda z\lambda^3 y - 2\lambda^3 x\lambda^2 z\lambda y + 12\lambda^2 x\lambda^2 z\lambda^2 y - 4\lambda x\lambda^4 z\lambda y = 0. \quad (3.17)$$

Subtracting (3.17) from (3.15) implies $\lambda^5 xz\lambda y - \lambda xz\lambda^5 y + \lambda^4 xz\lambda^2 y$

$$- \lambda^2 xz\lambda^4 y - 2\lambda x\lambda z\lambda^4 y + 2\lambda^4 x\lambda z\lambda y - 2\lambda^2 x\lambda z\lambda^3 y + 2\lambda^3 x\lambda z\lambda^2 y - 2\lambda x\lambda^2 z\lambda^3 y$$

$$+ 2\lambda^3 x\lambda^2 z\lambda y = 0.$$

Replacing x by λx and y by λy yields $\lambda^6 xz\lambda^2 y - \lambda^2 xz\lambda^6 y + \lambda^5 xz\lambda^3 y$

$- \lambda^3 xz\lambda^5 y - 2\lambda^2 x\lambda z\lambda^5 y + 2\lambda^5 x\lambda z\lambda^2 y - 2\lambda^3 x\lambda z\lambda^4 y + 2\lambda^4 x\lambda z\lambda^3 y$

$- 2\lambda^2 x\lambda^2 z\lambda^4 y + 2\lambda^4 x\lambda^2 z\lambda^2 y = 0$

$$\Rightarrow \quad 3\lambda^6 xz\lambda^2 y - 3\lambda^2 xz\lambda^6 y + 3\lambda^5 xz\lambda^3 y - 3\lambda^3 xz\lambda^5 y - 6\lambda^2 x\lambda z\lambda^5 y + 6\lambda^5 x\lambda z\lambda^2 y$$

$$- \; 6\lambda^3 x\lambda z\lambda^4 y + 6\lambda^4 x\lambda z\lambda^3 y - 6\lambda^2 x\lambda^2 z\lambda^4 y + 6\lambda^4 x\lambda^2 z\lambda^2 y = 0. \qquad (3.18)$$

Replacing x by λx in (3.14) yields $\lambda^6 xz\lambda y + 2\lambda^5 xz\lambda^2 y + 2\lambda^4 xz\lambda^3 y$

$+ \lambda^3 xz\lambda^4 y + 2\lambda^2 x\lambda z\lambda^4 y + 6\lambda^3 x\lambda z\lambda^3 y + 8\lambda^4 x\lambda z\lambda^2 y + 6\lambda^2 x\lambda^2 z\lambda^3 y$

$+ 12\lambda^3 x\lambda^2 z\lambda^2 y + 8\lambda^2 x\lambda^3 z\lambda^2 y = 0.$

Replacing y by λy in (3.16) yields $\lambda xz\lambda^6 y + \lambda^4 xz\lambda^3 y + 2\lambda^3 xz\lambda^4 y$

$+ 2\lambda^2 xz\lambda^5 y + 2\lambda^4 x\lambda z\lambda^2 y + 6\lambda^3 x\lambda z\lambda^3 y + 8\lambda^2 x\lambda z\lambda^4 y + 6\lambda^3 x\lambda^2 z\lambda^2 y$

$+ 12\lambda^2 x\lambda^2 z\lambda^3 y + 8\lambda^2 x\lambda^3 z\lambda^2 y = 0.$

Subtracting the last equation from the one preceding it we get

$$\lambda^6 xz\lambda y - \lambda xz\lambda^6 y + 2\lambda^5 xz\lambda^2 y - 2\lambda^2 xz\lambda^5 y + \lambda^4 xz\lambda^3 y - \lambda^3 xz\lambda^4 y$$

$$- 6\lambda^2 x\lambda z\lambda^4 y + 6\lambda^4 x\lambda z\lambda^2 y - 6\lambda^2 x\lambda^2 z\lambda^3 y + 6\lambda^3 x\lambda^2 z\lambda^2 y = 0.$$

Taking λ of both sides implies $\lambda^7 xz\lambda y + \lambda^6 xz\lambda^2 y - \lambda^2 xz\lambda^6 y - \lambda xz\lambda^7 y$

$+ 2\lambda^6 xz\lambda^2 y + 2\lambda^5 xz\lambda^3 y - 2\lambda^3 xz\lambda^5 y - 2\lambda^2 xz\lambda^6 y + \lambda^5 xz\lambda^3 y + \lambda^4 xz\lambda^4 y$

$- \lambda^4 xz\lambda^4 y - \lambda^3 xz\lambda^5 y - 6\lambda^3 x\lambda z\lambda^4 y - 6\lambda^2 x\lambda z\lambda^5 y + 6\lambda^5 x\lambda z\lambda^2 y$

$+ 6\lambda^4 x\lambda z\lambda^3 y - 6\lambda^3 x\lambda^2 z\lambda^3 y - 6\lambda^2 x\lambda^2 z\lambda^4 y + 6\lambda^4 x\lambda^2 z\lambda^2 y + 6\lambda^3 x\lambda^2 z\lambda^3 y = 0$

$$\Rightarrow \quad \lambda^7 xz\lambda y + 3\lambda^6 xz\lambda^2 y + 3\lambda^5 xz\lambda^3 y - 3\lambda^3 xz\lambda^5 y - 3\lambda^2 xz\lambda^6 y - \lambda xz\lambda^7 y$$

$$- 6\lambda^3 x\lambda z\lambda^4 y - 6\lambda^2 x\lambda z\lambda^5 y + 6\lambda^5 x\lambda z\lambda^2 y + 6\lambda^4 x\lambda z\lambda^3 y - 6\lambda^2 x\lambda^2 z\lambda^4 y$$

$$+ 6\lambda^4 x\lambda^2 z\lambda^2 y = 0. \qquad (3.19)$$

Subtracting (3.18) from (3.19) implies $\lambda^7 xz\lambda y - \lambda xz\lambda^7 y = 0$. Again apply Lemma 3.8 and the theorem is complete.

In Theorems 3.9, 3.10, and 3.11 we assumed that λ and λ^n were derivations of a prime ring R for n = 3,4, and 5, respectively. For n = 6 we will need the following special lemma.

Lemma 3.12 Let λ be a derivation of a ring R. Assume $\forall$ x,y,z $\in$ R, $\exists$ d, $c_i \in Z \setminus \{0\}$ such that $d(\lambda^n x\lambda z\lambda^{n+2} y - \lambda^{n+2} x\lambda z\lambda^n y)$

$$+ \sum_{i=1}^{t} c_i(\lambda^{a_i} xz\lambda^{b_i} y - \lambda^{b_i} xz\lambda^{a_i} y) = 0, \tag{3.20}$$

where $n \geq 2$, $a_i, b_i \in Z^+$, a_i distinct, $a_i < b_i$, $a_i + b_i = 2n + 3$, and $a_1 = 1$. Also assume $\forall$ x,y,z $\in$ R, $\exists$ $c_{ji} \in Z \setminus \{0\}$ such that

$$\sum_{i=1}^{t_o} c_{oi}(\lambda^{a_{oi}} xz\lambda^{b_{oi}} y - \lambda^{b_{oi}} xz\lambda^{a_{oi}} y) + \sum_{i=1}^{t_1} c_{1i}(\lambda^{a_{1i}} x\lambda z\lambda^{b_{1i}} y - \lambda^{b_{1i}} x\lambda z\lambda^{a_{1i}} y)$$

$$+ \dots + \sum_{i=1}^{t_k} c_{ki}(\lambda^{a_{ki}} x\lambda^k z\lambda^{b_{ki}} y - \lambda^{b_{ki}} x\lambda^k z\lambda^{a_{ki}} y) = 0 \tag{3.21}$$

where a_{ji}, $b_{ji} \in Z^+$, a_{ji} distinct for fixed j, $a_{ji} < b_{ji}$, $a_{ji} + b_{ji} = m - j$ for some fixed $m \in Z^+$, $m \geq k + 3$, and at least one $a_{ki} < n$. Then $\exists$ $c_i^* \in Z \setminus \{0\}$ and a_i^*, $b_i^* \in Z^+$ such that $\forall$ x,y,z $\in$ R,

$$\sum_{i=1}^{t^*} c_i^*(\lambda^{a_i^*} xz\lambda^{b_i^*} y - \lambda^{b_i^*} xz\lambda^{a_i^*} y) = 0.$$

Proof. Assume $a_{kh} = \min_i \{a_{ki}\}$ and let $n - a_{kh} = \ell > 0$.

Replacing x by $\lambda^\ell x$ and y by $\lambda^\ell y$ in (3.21) yields

$$\sum_{j=0}^{k}\sum_{i=1}^{t_j} c_{ji}\,(\lambda^{a'_{ji}}x\lambda^{j}z\lambda^{b'_{ji}}y - \lambda^{b'_{ji}}x\lambda^{j}z\lambda^{a'_{ji}}y) = 0, \qquad (3.21')$$

where $a'_{ji} = a_{ji} + \ell$ and $b'_{ji} = b_{ji} + \ell$, $\forall\ i,j$. Therefore $a'_{ji}, b'_{ji} \geq 2$, $\forall\ i,j$, and $a'_{ki}, b'_{ki} > n$, $\forall\ i$, except $a'_{kh} = n$.

<u>Case 1</u>: If $b'_{kh} - a'_{kh} = b'_{kh} - n = 1$, then $t_k = 1$. Therefore (3.21') becomes

$$\sum_{j=0}^{k-1}\sum_{i=1}^{t_j} c_{ji}(\lambda^{a'_{ji}}x\lambda^{j}z\lambda^{b'_{ji}}y - \lambda^{b'_{ji}}x\lambda^{j}z\lambda^{a'_{ji}}y) + c_{k1}(\lambda^{n}x\lambda^{k}z\lambda^{n+1}y - \lambda^{n+1}x\lambda^{k}z\lambda^{n}y) = 0.$$

Taking λ of both sides implies

$$\begin{aligned}
&\sum_{j=0}^{k-1}\sum_{i=1}^{t_j} c_{ji}(\lambda^{a'_{ji}+1}x\lambda^{j}z\lambda^{b'_{ji}}y + \lambda^{a'_{ji}}x\lambda^{j+1}z\lambda^{b'_{ji}}y + \lambda^{a'_{ji}}x\lambda^{j}z\lambda^{b'_{ji}+1}y \\
&\qquad - \lambda^{b'_{ji}+1}x\lambda^{j}z\lambda^{a'_{ji}}y - \lambda^{b'_{ji}}x\lambda^{j+1}z\lambda^{a'_{ji}}y - \lambda^{b'_{ji}}x\lambda^{j}z\lambda^{a'_{ji}+1}y) \\
&\quad + c_{k1}(\lambda^{n+1}x\lambda^{k}z\lambda^{n+1}y + \lambda^{n}x\lambda^{k+1}z\lambda^{n+1}y + \lambda^{n}x\lambda^{k}z\lambda^{n+2}y \\
&\quad - \lambda^{n+2}x\lambda^{k}z\lambda^{n}y - \lambda^{n+1}x\lambda^{k+1}z\lambda^{n}y - \lambda^{n+1}x\lambda^{k}z\lambda^{n+1}y) = 0 \\
\Rightarrow\ &\sum_{j=0}^{k-1}\sum_{i=1}^{t_j} c_{ji}\,(\lambda^{a'_{ji}+1}x\lambda^{j}z\lambda^{b'_{ji}}y + \lambda^{a'_{ji}}x\lambda^{j}z\lambda^{b'_{ji}+1}y - \lambda^{b'_{ji}+1}x\lambda^{j}z\lambda^{a'_{ji}}y \\
&\quad - \lambda^{b'_{ji}}x\lambda^{j}z\lambda^{a'_{ji}+1}) + c_{k1}\,(\lambda^{n}x\lambda^{k}z\lambda^{n+2}y - \lambda^{n+2}x\lambda^{k}z\lambda^{n}y) = 0.
\end{aligned}$$

Now substitute $\lambda^{k-1}z$ for z in (3.20) and use the resulting equation (with the coefficients appropriately adjusted) to replace the term involving $\lambda^{k}z$ in the last equation by terms involving $\lambda^{k-1}z$.

<u>Case 2</u>: If $b'_{kh} - a'_{kh} = b'_{kh} - n = 2$, then $t_k = 1$. Therefore (3.21') becomes

$$\sum_{j=0}^{k-1}\sum_{i=1}^{t_j} c_{ji}(\lambda^{a'_{ji}}x\lambda^{j}z\lambda^{b'_{ji}}y - \lambda^{b'_{ji}}x\lambda^{j}z\lambda^{a'_{ji}}y)$$

$$+ \; c_{k1}(\lambda^{n}x\lambda^{k}z\lambda^{n+2}y - \lambda^{n+2}x\lambda^{k}z\lambda^{n}y) = 0.$$

As in Case 1, use (3.20) to replace the last term with terms involving $\lambda^{k-1}z$.

<u>Case 3</u>: If $b'_{kh} - a'_{kh} = b'_{kh} - n = r > 2$, then define $a''_{ki} = a'_{ki}$, $i \neq h$, and $b''_{ki} = b'_{ki}$, $i \neq h$, and use (3.20) to replace a'_{kh} with $a''_{kh} = a'_{kh} + 2$ and b'_{kh} with $b''_{kh} = b'_{kh} - 2$. We may do this since (3.20) implies

$$(c_{kh}d)\lambda^{a'_{kh}}x\lambda^{k}z\lambda^{b'_{kh}}y = c_{kh}\,[d\lambda^{a'_{kh}+2}x\lambda^{k}z\lambda^{b'_{kh}-2}y -$$

$$\sum_{i=1}^{t} c_i(\lambda^{a_i}x\lambda^{k-1}z\lambda^{b_i+r-2}y - \lambda^{b_i}x\lambda^{k-1}z\lambda^{a_i+r-2}x)]$$

and

$$(c_{kh}d)\lambda^{b'_{kh}}x\lambda^{k}z\lambda^{a'_{kh}}y = c_{kh}[d\lambda^{b'_{kh}-2}x\lambda^{k}z\lambda^{a'_{kh}+2}y +$$

$$\sum_{i=1}^{t} c_i(\lambda^{a_i+r-2}x\lambda^{k-1}z\lambda^{b_i}y - \lambda^{b_i+r-2}x\lambda^{k-1}z\lambda^{a_i}y)].$$

Notice that in all three cases we must introduce a $\lambda^{k-1}z$ term of the form $c(\lambda x\lambda^{k-1}z\lambda^{b}y - \lambda^{b}x\lambda^{k-1}z\lambda y)$. Moreover this term does not cancel with any other term involving $\lambda^{k-1}z$ since $a'_{ji} \geq 2$ $\forall$ i,j. If we are dealing with case three, note that we now have $a''_{ki}, b''_{ki} \geq n + 1$, $\forall$ i. We continue using (3.20) to eliminate expressions with $\lambda^{k}z$.

Since a''_{ki}, $b''_{ki} \geq n + 1$, $\forall$ i, as we eliminate an expression involving $\lambda^k z$, we can only get expressions involving $\lambda^{k-1}z$ of the form $c(\lambda^a x\lambda^{k-1}z\lambda^b y - \lambda^b x\lambda^{k-1}z\lambda^a y)$ where a, b $\geq$ 2. The conclusion is that after getting rid of all $\lambda^k z$ terms, there must still be a $\lambda^{k-1}z$ term of the form $c(\lambda x\lambda^{k-1}z\lambda^b y - \lambda^b x\lambda^{k-1}z\lambda y)$. We now start the whole process over, eliminating all terms with $\lambda^{k-1}z$. Eventually we must arrive at the desired conclusion.

<u>Theorem 3.13</u> If λ and λ^6 are derivations of a prime ring R and characteristic R is sufficiently large, then λ is algebraic.

<u>Proof.</u> $\lambda^6(xy) = \sum_{i=0}^{6} \binom{6}{i} \lambda^{6-i}x\lambda^i y = \lambda^6 xy + x\lambda^6 y$, $\forall$ x, y $\in$ R,

$$\Rightarrow \quad 6\lambda^5 x\lambda y + 15\lambda^4 x\lambda^2 y + 20\lambda^3 x\lambda^3 y + 15\lambda^2 x\lambda^4 y + 6\lambda x\lambda^5 y = 0. \qquad (3.22)$$

Replacing x by λx in (3.22) yields $6\lambda^6 x\lambda y + 15\lambda^5 x\lambda^2 y + 20\lambda^4 x\lambda^3 y + 15\lambda^3 x\lambda^4 y + 6\lambda^2 x\lambda^5 y = 0.$

Replacing x by xz yields $6\lambda^6 xz\lambda y + 6x\lambda^6 z\lambda y + 15\lambda^5 xz\lambda^2 y + 75\lambda^4 x\lambda z\lambda^2 y$
$+ 150\lambda^3 x\lambda^2 z\lambda^2 y + 150\lambda^2 x\lambda^3 z\lambda^2 y + 75\lambda x\lambda^4 z\lambda^2 y + 15x\lambda^5 z\lambda^2 y + 20\lambda^4 xz\lambda^3 y$
$+ 80\lambda^3 x\lambda z\lambda^3 y + 120\lambda^2 x\lambda^2 z\lambda^3 y + 80\lambda x\lambda^3 z\lambda^3 y + 20x\lambda^4 z\lambda^3 y + 15\lambda^3 xz\lambda^4 y$
$+ 45\lambda^2 x\lambda z\lambda^4 y + 45\lambda x\lambda^2 z\lambda^4 y + 15x\lambda^3 z\lambda^4 y + 6\lambda^2 xz\lambda^5 y + 12\lambda x\lambda z\lambda^5 y$
$+ 6x\lambda^2 z\lambda^5 y = 0$

$$\Rightarrow \quad 6\lambda^6 xz\lambda y + 15\lambda^5 xz\lambda^2 y + 20\lambda^4 xz\lambda^3 y + 15\lambda^3 xz\lambda^4 y + 6\lambda^2 xz\lambda^5 y$$

$$+ 75\lambda^4 x\lambda z\lambda^2 y + 150\lambda^3 x\lambda^2 z\lambda^2 y + 150\lambda^2 x\lambda^3 z\lambda^2 y + 75\lambda x\lambda^4 z\lambda^2 y + 80\lambda^3 x\lambda z\lambda^3 y$$

$$+ 120\lambda^2 x\lambda^2 z\lambda^3 y + 80\lambda x\lambda^3 z\lambda^3 y + 45\lambda^2 x\lambda z\lambda^4 y + 45\lambda x\lambda^2 z\lambda^4 y$$

$$+ 12\lambda x\lambda z\lambda^5 y = 0. \tag{3.23}$$

By (3.22), $75\lambda x\lambda^4 z\lambda^2 y = -\,30\lambda x\lambda^5 z\lambda y - 100\lambda x\lambda^3 z\lambda^3 y - 75\lambda x\lambda^2 z\lambda^4 y$

$- 30\lambda x\lambda z\lambda^5 y$

$$\Rightarrow \quad 6\lambda^6 xz\lambda y + 15\lambda^5 xz\lambda^2 y + 20\lambda^4 xz\lambda^3 y + 15\lambda^3 xz\lambda^4 y + 6\lambda^2 xz\lambda^5 y$$

$$+ 75\lambda^4 x\lambda z\lambda^2 y + 150\lambda^3 x\lambda^2 z\lambda^2 y + 150\lambda^2 x\lambda^3 z\lambda^2 y - 30\lambda x\lambda^5 z\lambda y + 80\lambda^3 x\lambda z\lambda^3 y$$

$$+ 120\lambda^2 x\lambda^2 z\lambda^3 y - 20\lambda x\lambda^3 z\lambda^3 y + 45\lambda^2 x\lambda z\lambda^4 y - 30\lambda x\lambda^2 z\lambda^4 y$$

$$- 18\lambda x\lambda z\lambda^5 y = 0. \tag{3.24}$$

Replacing y by λy in (3.22) yields $6\lambda^5 x\lambda^2 y + 15\lambda^4 x\lambda^3 y + 20\lambda^3 x\lambda^4 y$

$+ 15\lambda^2 x\lambda^5 y + 6\lambda x\lambda^6 y = 0.$

Replacing y by zy yields $6\lambda^5 x\lambda^2 zy + 12\lambda^5 x\lambda z\lambda y + 6\lambda^5 xz\lambda^2 y$

$$+ 15\lambda^4 x\lambda^3 zy + 45\lambda^4 x\lambda^2 z\lambda y + 45\lambda^4 x\lambda z\lambda^2 y + 15\lambda^4 xz\lambda^3 y + 20\lambda^3 x\lambda^4 zy$$

$$+ 80\lambda^3 x\lambda^3 z\lambda y + 120\lambda^3 x\lambda^2 z\lambda^2 y + 80\lambda^3 x\lambda z\lambda^3 y + 20\lambda^3 xz\lambda^4 y + 15\lambda^2 x\lambda^5 zy$$

$$+ 75\lambda^2 x\lambda^4 z\lambda y + 150\lambda^2 x\lambda^3 z\lambda^2 y + 150\lambda^2 x\lambda^2 z\lambda^3 y + 75\lambda^2 x\lambda z\lambda^4 y + 15\lambda^2 xz\lambda^5 y$$

$$+ 6\lambda x\lambda^6 zy + 6\lambda xz\lambda^6 y = 0$$

$$\Rightarrow \quad 6\lambda^5 xz\lambda^2 y + 15\lambda^4 xz\lambda^3 y + 20\lambda^3 xz\lambda^4 y + 15\lambda^2 xz\lambda^5 y + 6\lambda xz\lambda^6 y$$

$$+ 12\lambda^5 x\lambda z\lambda y + 45\lambda^4 x\lambda^2 z\lambda y + 45\lambda^4 x\lambda z\lambda^2 y + 80\lambda^3 x\lambda^3 z\lambda y + 120\lambda^3 x\lambda^2 z\lambda^2 y$$

$$+ 80\lambda^3 x\lambda z\lambda^3 y + 75\lambda^2 x\lambda^4 z\lambda y + 150\lambda^2 x\lambda^3 z\lambda^2 y + 150\lambda^2 x\lambda^2 z\lambda^3 y$$

$$+ 75\lambda^2 x\lambda z\lambda^4 y = 0. \tag{3.25}$$

By (3.22), $75\lambda^2 x\lambda^4 z\lambda y = - 30\lambda x\lambda^5 z\lambda y - 100\lambda^3 x\lambda^3 z\lambda y - 75\lambda^4 x\lambda^2 z\lambda y - 30\lambda^5 x\lambda z\lambda y$

$$\Rightarrow \quad 6\lambda^5 xz\lambda^2 y + 15\lambda^4 xz\lambda^3 y + 20\lambda^3 xz\lambda^4 y + 15\lambda^2 xz\lambda^5 y + 6\lambda xz\lambda^6 y$$

$$- 18\lambda^5 x\lambda z\lambda y - 30\lambda^4 x\lambda^2 z\lambda y + 45\lambda^4 x\lambda z\lambda^2 y - 20\lambda^3 x\lambda^3 z\lambda y + 120\lambda^3 x\lambda^2 z\lambda^2 y$$

$$+ 80\lambda^3 x\lambda z\lambda^3 y - 30\lambda x\lambda^5 z\lambda y + 150\lambda^2 x\lambda^3 z\lambda^2 y + 150\lambda^2 x\lambda^2 z\lambda^3 y$$

$$+ 75\lambda^2 x\lambda z\lambda^4 y = 0. \tag{3.26}$$

Subtracting (3.26) from (3.24) implies $6\lambda^6 xz\lambda y + 9\lambda^5 xz\lambda^2 y$

$$+ 5\lambda^4 xz\lambda^3 y - 5\lambda^3 xz\lambda^4 y - 9\lambda^2 xz\lambda^5 y - 6\lambda xz\lambda^6 y + 30\lambda^4 x\lambda z\lambda^2 y$$

$$+ 30\lambda^3 x\lambda^2 z\lambda^2 y - 30\lambda^2 x\lambda^2 z\lambda^3 y - 20\lambda x\lambda^3 z\lambda^3 y + 20\lambda^3 x\lambda^3 z\lambda y$$

$$- 30\lambda^2 x\lambda z\lambda^4 y - 30\lambda x\lambda^2 z\lambda^4 y + 30\lambda^4 x\lambda^2 z\lambda y - 18\lambda x\lambda z\lambda^5 y + 18\lambda^5 x\lambda z\lambda y = 0.$$

Replacing x by λx and y by λy and multiplying by 7 yields

$$42\lambda^7 xz\lambda^2 y + 63\lambda^6 xz\lambda^3 y + 35\lambda^5 xz\lambda^4 y - 35\lambda^4 xz\lambda^5 y - 63\lambda^3 xz\lambda^6 y$$

$$- 42\lambda^2 xz\lambda^7 y + 210\lambda^5 x\lambda z\lambda^3 y + 210\lambda^4 x\lambda^2 z\lambda^3 y - 210\lambda^3 x\lambda^2 z\lambda^4 y$$

$$- 140\lambda^2 x\lambda^3 z\lambda^4 y + 140\lambda^4 x\lambda^3 z\lambda^2 y - 210\lambda^3 x\lambda z\lambda^5 y - 210\lambda^2 x\lambda^2 z\lambda^5 y$$

$$+ 210\lambda^5 x\lambda^2 z\lambda^2 y - 126\lambda^2 x\lambda z\lambda^6 y + 126\lambda^6 x\lambda z\lambda^2 y = 0. \tag{3.27}$$

Replacing x by λx in (3.23) yields $6\lambda^7 xz\lambda y + 15\lambda^6 xz\lambda^2 y + 20\lambda^5 xz\lambda^3 y$

$$+ 15\lambda^4 xz\lambda^4 y + 6\lambda^3 xz\lambda^5 y + 75\lambda^5 x\lambda z\lambda^2 y + 150\lambda^4 x\lambda^2 z\lambda^2 y + 150\lambda^3 x\lambda^3 z\lambda^2 y$$

$$+ 75\lambda^2 x\lambda^4 z\lambda^2 y + 80\lambda^4 x\lambda z\lambda^3 y + 120\lambda^3 x\lambda^2 z\lambda^3 y + 80\lambda^2 x\lambda^3 z\lambda^3 y$$

$$+ 45\lambda^3 x\lambda z\lambda^4 y + 45\lambda^2 x\lambda^2 z\lambda^4 y + 12\lambda^2 x\lambda z\lambda^5 y = 0.$$

Replacing y by λy in (3.23) yields $6\lambda^5 xz\lambda^3 y + 15\lambda^4 xz\lambda^4 y + 20\lambda^3 xz\lambda^5 y$

$$+ 15\lambda^2 xz\lambda^6 y + 6\lambda xz\lambda^7 y + 12\lambda^5 x\lambda z\lambda^2 y + 45\lambda^4 x\lambda^2 z\lambda^2 y + 45\lambda^4 x\lambda z\lambda^3 y$$

$$+ 80\lambda^3 x\lambda^3 z\lambda^2 y + 120\lambda^3 x\lambda^2 z\lambda^3 y + 80\lambda^3 x\lambda z\lambda^4 y + 75\lambda^2 x\lambda^4 z\lambda^2 y + 150\lambda^2 x\lambda^3 z\lambda^3 y$$

$$+ 150\lambda^2 x\lambda^2 z\lambda^4 y + 75\lambda^2 x\lambda z\lambda^5 y = 0.$$

Subtracting the last equation from the one preceding it we get

$$6\lambda^7 xz\lambda y + 15\lambda^6 xz\lambda^2 y + 14\lambda^5 xz\lambda^3 y - 14\lambda^3 xz\lambda^5 y - 15\lambda^2 xz\lambda^6 y - 6\lambda xz\lambda^7 y$$

$$+ 63\lambda^5 x\lambda z\lambda^2 y + 105\lambda^4 x\lambda^2 z\lambda^2 y + 70\lambda^3 x\lambda^3 z\lambda^2 y + 35\lambda^4 x\lambda z\lambda^3 y$$

$$- 70\lambda^2 x\lambda^3 z\lambda^3 y - 35\lambda^3 x\lambda z\lambda^4 y - 105\lambda^2 x\lambda^2 z\lambda^4 y - 63\lambda^2 x\lambda z\lambda^5 y = 0.$$

Taking λ of both sides implies $6\lambda^8 xz\lambda y + 6\lambda^7 xz\lambda^2 y + 15\lambda^7 xz\lambda^2 y$

$$+ 15\lambda^6 xz\lambda^3 y + 14\lambda^6 xz\lambda^3 y + 14\lambda^5 xz\lambda^4 y - 14\lambda^4 xz\lambda^5 y - 14\lambda^3 xz\lambda^6 y$$

$$- 15\lambda^3 xz\lambda^6 y - 15\lambda^2 xz\lambda^7 y - 6\lambda^2 xz\lambda^7 y - 6\lambda xz\lambda^8 y + 63\lambda^6 x\lambda z\lambda^2 y$$

$$+ 63\lambda^5 x\lambda z\lambda^3 y + 105\lambda^5 x\lambda^2 z\lambda^2 y + 105\lambda^4 x\lambda^2 z\lambda^3 y + 70\lambda^4 x\lambda^3 z\lambda^2 y$$

$$+ 70\lambda^3 x\lambda^3 z\lambda^3 y + 35\lambda^5 x\lambda z\lambda^3 y + 35\lambda^4 x\lambda z\lambda^4 y - 70\lambda^3 x\lambda^3 z\lambda^3 y - 70\lambda^2 x\lambda^3 z\lambda^4 y$$

$$- 35\lambda^4 x\lambda z\lambda^4 y - 35\lambda^3 x\lambda z\lambda^5 y - 105\lambda^3 x\lambda^2 z\lambda^4 y - 105\lambda^2 x\lambda^2 z\lambda^5 y$$

$$- 63\lambda^3 x\lambda z\lambda^5 y - 63\lambda^2 x\lambda z\lambda^6 y = 0.$$

Combining terms and multiplying by 2 we get $12\lambda^8 xz\lambda y + 42\lambda^7 xz\lambda^2 y$

$$+ 58\lambda^6 xz\lambda^3 y + 28\lambda^5 xz\lambda^4 y - 28\lambda^4 xz\lambda^5 y - 58\lambda^3 xz\lambda^6 y - 42\lambda^2 xz\lambda^7 y$$

$$- 12\lambda xz\lambda^8y + 126\lambda^6x\lambda z\lambda^2y + 196\lambda^5x\lambda z\lambda^3y + 210\lambda^5x\lambda^2z\lambda^2y$$
$$+ 210\lambda^4x\lambda^2z\lambda^3y + 140\lambda^4x\lambda^3z\lambda^2y - 140\lambda^2x\lambda^3z\lambda^4y - 196\lambda^3x\lambda z\lambda^5y$$
$$- 210\lambda^3x\lambda^2z\lambda^4y - 210\lambda^2x\lambda^2z\lambda^5y - 126\lambda^2x\lambda z\lambda^6y = 0. \tag{3.28}$$

Subtracting (3.27) from (3.28) implies $12\lambda^8xz\lambda y - 5\lambda^6xz\lambda^3y$

$$- 7\lambda^5xz\lambda^4y + 7\lambda^4xz\lambda^5y + 5\lambda^3xz\lambda^6y - 12\lambda xz\lambda^8y - 14\lambda^5x\lambda z\lambda^3y$$
$$+ 14\lambda^3x\lambda z\lambda^5y = 0.$$

This equation and (3.28) satisfy the hypotheses of Lemma 3.12. Therefore $\exists\, c_i^* \in Z \setminus \{0\}$ and $a_i^*, b_i^* \in Z^+$ such that

$$\sum_{i=1}^{t^*} c_i^* (\lambda^{a_i^*}xz\lambda^{b_i^*}y - \lambda^{b_i^*}xz\lambda^{a_i^*}y) = 0.$$

By Lemma 3.8 we conclude that λ is algebraic.

3.4 Results for Arbitrary $n \in Z^+$

The last two theorems of this chapter concern the situation where λ and λ^n are derivations of a prime ring R for general $n \in Z^+$. Here and in Chapters 4 and 5 we will use the following simple but versatile lemma.

Lemma 3.14 Assume λ is a derivation of a prime ring R and $\exists$ $0 \neq a \in R$ such that $a(\lambda^nR) = 0$ or $(\lambda^nR)a = 0$. Then $\lambda^{2n-1} = 0$.

Proof. Assuming $a(\lambda^nR) = 0$ and $x, y \in R$, we have $a\lambda^n(xy) = 0$

$$\Rightarrow \quad a\left(\sum_{i=0}^{n} \binom{n}{i} \lambda^i x \lambda^{n-i} y\right) = 0 \tag{3.29}$$

Replacing x by $\lambda^{n-1}x$ yields

$$a\lambda^{n-1}x\lambda^{n}y = 0 \tag{3.30}$$

Replacing x by $\lambda^{n-2}x$ and y by λy in (3.29) and using (3.30) yields

$$a\lambda^{n-2}x\lambda^{n+1}y = 0 \tag{3.31}$$

Replacing x by $\lambda^{n-3}x$ and y by $\lambda^{2}y$ in (3.29) and using (3.30) and (3.31) yields

$$a\lambda^{n-3}x\lambda^{n+2}y = 0$$

Continuing this process we eventually obtain $ax\lambda^{2n-1}y = 0$. Since this is true $\forall$ x, y $\in$ R, by the primeness of R we conclude that $\lambda^{2n-1} = 0$. Similarly, if $0 \neq a \in R$ and $(\lambda^{n}R)a = 0$ then $\lambda^{2n-1} = 0$.

<u>Theorem 3.15</u> If λ and λ^{n} are derivations of a prime ring R and $\exists$ $a \in R$ such that $\lambda a \neq 0$ and $\lambda^{2}a = 0$, then $\lambda^{2n-3} = 0$. If in addition characteristic $R \neq 2$, then $\lambda^{n} = 0$ if n is odd and $\lambda^{n-1} = 0$ if n is even.

<u>Proof.</u> λ^{n} is a derivation implies $\lambda^{n}(xy) = \lambda^{n}xy + x\lambda^{n}y$

$$= \sum_{i=0}^{n} \binom{n}{i} \lambda^{n-i}x\lambda^{i}y, \ \forall\, x,y \in R.$$

$$\text{Therefore } \sum_{i=1}^{n-1} \binom{n}{i} \lambda^{n-i}x\lambda^{i}y = 0, \quad \forall\, x,y \in R.$$

Letting y = a implies $$\lambda^{n-1}x\lambda a = 0, \quad \forall\, x \in R \tag{3.32}$$

Letting x = a implies $$\lambda a\lambda^{n-1}y = 0, \quad \forall\, y \in R \tag{3.33}$$

In either case, $\lambda^{2n-3} = 0$ by Lemma 3.14. If characteristic $R \neq 2$, then by Posner's Proposition 3.1, $(\lambda^n)^2 = 0 \Rightarrow \lambda^n = 0$. Proposition 2.5 says the index of nilpotency must be an odd number and we have $\lambda^{n-1} = 0$ if n is even. If characteristic $R = 0$, then an alternative proof is available by noting that (3.32) and (3.33) together imply $\lambda^{2n-3}(R\lambda aR) = 0$. Therefore by Proposition 2.8, $\lambda^{2n-3} = 0$.

Theorem 3.16 Assume λ and λ^n are derivations of a prime ring R, Λ annihilates C, and characteristic $R = 0$. If $\exists\ 0 \neq a \in R$ and $0 \neq c \in C$ such that $\lambda a = ca$, then λ is algebraic.

Proof. λ^n is a derivation implies $\sum_{i=1}^{n-1} \binom{n}{i} \lambda^{n-i}x\lambda^i y = 0$, $\forall\ x,y \in R$.

Letting x = a and using $\Lambda(C) = 0$ implies $a\left(\sum_{i=1}^{n-1} \binom{n}{i} c^{n-i}\lambda^i\right)y = 0$, $\forall\ y \in R$.

Letting y = a and using $\Lambda(C) = 0$ implies $[(\sum_{i=1}^{n-1} \binom{n}{i} c^{n-i}\lambda^i)x]a = 0$, $\forall\ x \in R$.

Assume $\sum_{i=1}^{n-1} \binom{n}{i} c^{n-i}\lambda^i = \prod_{i=1}^{t} (\lambda - c_i)^{n_i}$, where the c_i are mutually distinct elements of F. Then $\left(\prod_{i=1}^{t} (\lambda - c_i)^{n_i}x\right) a = 0$

$= a\left(\prod_{i=1}^{t} (\lambda - c_i)^{n_i}x\right)$, $\forall\ x \in R$.

Since $\prod_{i=1, i\neq j}^{t} (\lambda - c_i)^{n_i}$, $j = 1,2, \ldots, t$, are relatively prime as polynomials of λ, $\exists$ polynomials $f_1, f_2, \ldots, f_t$ such that

$$1 = f_1(\lambda) \prod_{i=2}^{t} (\lambda - c_i)^{n_i} + f_2(\lambda) \prod_{i=1, i\neq 2}^{t} (\lambda - c_i)^{n_i}$$

$$+\ldots+ \; f_t(\lambda) \prod_{i=1}^{t-1} (\lambda - c_i)^{n_i} \quad .$$

If we define $\bar{R}_{c_i} = \{x \in R \mid [(\lambda - c_i)^m x]\, a = 0 \text{ for some } m\}$ and $\bar{\bar{R}}_{c_j} = \{x \in R \mid a(\lambda - c_j)^m x = 0 \text{ for some } m\}$, then $R = \sum_{i=1}^{t} \bar{R}_{c_i}$ and $R = \sum_{j=1}^{t} \bar{\bar{R}}_{c_j}$. Consider $RaR = (\sum_{i=1}^{t} \bar{R}_{c_i})\, a\, (\sum_{j=1}^{t} \bar{\bar{R}}_{c_j})$. For all $z \in \bar{R}_{c_i}$ and for all $w \in \bar{\bar{R}}_{c_j}$, $\exists\; N_{ij} \in Z^+$ such that $(\lambda - (c_i + c_j + c))^{2N_{ij}}(zaw)$

$$= \sum_{k=0}^{2N_{ij}} \binom{2N_{ij}}{k} (\lambda - c_i)^{2N_{ij}-k}\, z\, a\, (\lambda - c_j)^k w = 0.$$ Therefore

$$\left(\sum_{i,j=1}^{t} (\lambda - (c_i + c_j + c))^{2N_{ij}} \right) (RaR) = 0.$$

By Proposition 2.8, λ is algebraic.

4. DERIVATIONS SATISFYING $f(\lambda,\delta) = \lambda^n\delta^m = 0$

Let λ and δ be derivations of a prime ring R. In this chapter we investigate what can be said when $\lambda^n\delta^m = 0$ for $n,m \in Z^+$. If characteristic $R \neq 2$, we know by Posner's Proposition 2.1 that $\lambda\delta = 0$ implies $\lambda = 0$ or $\delta = 0$. Without the characteristic restriction we can still say the following:

<u>Lemma 4.1</u> If $\lambda\delta = 0$, then either $\lambda = 0$ or $\delta^2 = 0$.

<u>Proof.</u> For any $x,y \in R$, we have $\lambda\delta(xy) = \lambda x\delta y + \delta x\lambda y = 0$.

Replacing x by δx we get $\delta^2 x\lambda y = 0$. Now use Lemma 3.14 to obtain either $\lambda = 0$ or $\delta^2 = 0$.

The distinction between Posner's result and Lemma 4.1 is made clear by a simple example. Consider the 2×2 matrix ring over the Galois field $GF(2^2) = \{0,1,w,w^2\}$, with λ and δ defined by

$$\lambda(X) = \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X\right] \text{ and } \delta(X) = \left[\begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}, X\right], \ \forall\ X \in (GF(2^2))_{2\times 2}$$

The characteristic of $GF(2^2) = 2$ and we have $\lambda\delta = 0$, $\lambda \neq 0$, $\delta \neq 0$, and $\delta^2 = 0$.

Given $\lambda\delta = 0$, Lemma 4.1 tells us that while we cannot insure either λ or δ is identically zero, we do know at least one of them is nilpotent. This leads us to conjecture that $\forall\ n,m \in Z^+$, $\lambda^n\delta^m = 0$ implies either λ is nilpotent or δ is nilpotent. We show that if $n = 1$ or $m = 1$, or if $\lambda\delta = \delta\lambda$, then this is indeed the case.

4.1 For $n = 1$ or $m = 1$

Theorem 4.2 If $\lambda\delta^m = 0$, then either $\lambda = 0$ or $\delta^r = 0$ where $r \leq 4m-1$.

Proof. Proceeding by induction, Lemma 4.1 implies that the result is true for $m = 1$. Assume the statement is true for $m = 1,2,\ldots,k-1$. If $\lambda\delta^k = 0$, then $\lambda\delta^k(xy) = 0$, $\forall\ x,y \in R$.

$$\Rightarrow \lambda\left(\sum_{i=0}^{k} \binom{k}{i} \delta^{k-i}x\delta^{i}y\right) = 0.$$

Replacing x by $\delta^{k-1}x$ and y by $\delta^k y$ yields $\lambda(\delta^{k-1}x\delta^{2k}y) = 0$

$$\Rightarrow \lambda\delta^{k-1}x\delta^{2k}y + \delta^{k-1}x\lambda\delta^{2k}y = 0$$

$$\Rightarrow \quad \lambda\delta^{k-1}x\delta^{2k}y = 0.$$

Applying Lemma 3.14 gives the desired conclusion.

Theorem 4.3 If $\lambda^n\delta = 0$, then either $\delta^2 = 0$ or $\lambda^r = 0$ where $r \leq 12n-9$.

Proof. We know that the derivations of R from a Lie ring under commutation [20]. Therefore $[\delta,\lambda] = \delta\lambda - \lambda\delta$ is a derivation

$$\Rightarrow [\delta\lambda-\lambda\delta,\lambda] = \delta\lambda^2 - 2\lambda\delta\lambda + \lambda^2\delta \text{ is a derivation}$$

$$\Rightarrow [\delta\lambda^2 - 2\lambda\delta\lambda + \lambda^2\delta,\lambda] = \delta\lambda^3 - 3\lambda\delta\lambda^2 + 3\lambda^2\delta\lambda - \lambda^3\delta \text{ is}$$

a derivation. Continuing we get

$$\sum_{i=0}^{2n-1} \binom{2n-1}{i} (-1)^i \lambda^i\delta\lambda^{2n-1-i} \text{ is a derivation.}$$

Supressing the coefficients and using $\lambda^n\delta = 0$ we get

$$\delta\lambda^{2n-1} + \lambda\delta\lambda^{2n-2} + \ldots + \lambda^{n-1}\delta\lambda^n \text{ is a derivation.}$$

Then applying Lemma 4.1 to $(\delta\lambda^{2n-1} + \lambda\delta\lambda^{2n-2} + \ldots + \lambda^{n-1}\delta\lambda^n)\delta = 0$ we have $\delta^2 = 0$ or

$$\delta\lambda^{2n-1} + \lambda\delta\lambda^{2n-2} + \ldots + \lambda^{n-1}\delta\lambda^n = 0. \tag{4.1}$$

If $\delta^2 \neq 0$, then we premultiply (4.1) by λ^{n-1} to get $\lambda^{n-1}\delta\lambda^{2n-1} = 0$. Premultiplying (4.1) by λ^{n-2} it follows that

$$\lambda^{n-2}\delta\lambda^{2n-1} + \lambda^{n-1}\delta\lambda^{2n-2} = 0$$

$$\Rightarrow \quad (\lambda^{n-2}\delta\lambda^{2n-1} + \lambda^{n-1}\delta\lambda^{2n-2})\lambda = 0$$

$$\Rightarrow \quad \lambda^{n-2}\delta\lambda^{2n} = 0.$$

Premultiplying (4.1) by λ^{n-3} it follows that

$$\lambda^{n-3}\delta\lambda^{2n-1} + \lambda^{n-2}\delta\lambda^{2n-2} + \lambda^{n-1}\delta\lambda^{2n-3} = 0$$

$$\Rightarrow \quad (\lambda^{n-3}\delta\lambda^{2n-1} + \lambda^{n-2}\delta\lambda^{2n-2} + \lambda^{n-1}\delta\lambda^{2n-3})\lambda^2 = 0$$

$$\Rightarrow \quad \lambda^{n-3}\delta\lambda^{2n+1} = 0.$$

Eventually we arrive at $\delta\lambda^{3n-2} = 0$. Applying Theorem 4.2 completes the proof.

4.2 In Case λ and δ Commute

Theorem 4.4 If $\lambda^n\delta^m = 0$ and $[\lambda,\delta] = 0$, then either λ is nilpotent or δ is nilpotent.

<u>Proof.</u> $\forall\, x,y \in R,\ \lambda^n\delta^m(\,xy\,) = 0$

$$\Rightarrow \quad 0 = \lambda^n\delta^m(\delta^{m-1}\lambda^n x\delta\lambda^{n-1}y)$$

$$= \lambda^n(\delta^{m-1}\lambda^n x\delta^{m+1}\lambda^{n-1}y$$

$$= \delta^{m-1}\lambda^{2n}x\delta^{m+1}\lambda^{n-1}y$$

$$\Rightarrow \quad 0 = \lambda^n\delta^m(\delta^{m-2}\lambda^{2n}x\delta^{m+1}\lambda^{n-1}y)$$

$$= \lambda^n(\delta^{m-2}\lambda^{2n}x\delta^{2m+1}\lambda^{n-1}y)$$

$$= \delta^{m-2}\lambda^{3n}x\delta^{2m+1}\lambda^{n-1}y$$

$$\Rightarrow \quad 0 = \lambda^n\delta^m(\delta^{m-3}\lambda^{3n}x\delta^{2m+1}\lambda^{n-1}y)$$

$$= \lambda^n(\delta^{m-3}\lambda^{3n}x\delta^{3m+1}\lambda^{n-1}y)$$

$$= \delta^{m-3}\lambda^{4n}x\delta^{3m+1}\lambda^{n-1}y$$

$$\vdots$$

$$\Rightarrow \quad 0 = \lambda^n\delta^m(\delta\lambda^{(m-1)n}x\delta^{(m-2)m+1}\lambda^{n-1}y)$$

$$= \lambda^n(\delta\lambda^{(m-1)n}x\delta^{(m-1)m+1}\lambda^{n-1}y)$$

$$= \delta\lambda^{mn}x\delta^{(m-1)m+1}\lambda^{n-1}y$$

$$\Rightarrow \quad 0 = \lambda^n\delta^m(\lambda^{mn}x\delta^{(m-1)m+1}\lambda^{n-1}y)$$

$$= \lambda^n(\lambda^{mn}x\delta^{m^2+1}\lambda^{n-1}y)$$

$$= \lambda^{(m+1)n}x\delta^{m^2+1}\lambda^{n-1}y$$

By Lemma 3.14, either $\lambda^{n-1}\delta^{m^2+1} = \lambda^{n-1}\delta^{m_1} = 0$ or $\lambda^{2(m+1)n-1} = 0$.

If $\lambda^{2(m+1)n-1} \neq 0$, we apply the above argument to $\lambda^{n-1}\delta^{m_1} = 0$ to get $\lambda^{n-2}\delta^{m_1^2+1} = \lambda^{n-2}\delta^{m_2} = 0$ or $\lambda^{2(m^2+2)(n-1)-1} = 0$. If λ is not nilpotent we continue this process to eventually get $\delta^{m_n} = 0$.

One should observe that if λ_1, λ_2, and λ_3 are derivations of a prime ring R and $\lambda_1^n \lambda_2^m \lambda_3^h = 0$, it does not follow that λ_i is nilpotent for at least one i. For example, let D be a division ring and consider $D_{3 \times 3}$. Assume λ_1, λ_2, and λ_3 are the inner derivations defined by the unit matrices E_{11}, E_{22}, and E_{33}, respectively. Then λ_i is obviously not nilpotent and $\lambda_i \lambda_j = \lambda_j \lambda_i$ for i,j = 1,2,3. For fixed i,

$$\lambda_i X = \sum_{k=1} x_{ik} E_{ik} - \sum_{\ell=1} x_{\ell i} E_{\ell i}, \quad \forall\, X = (x_{ij}) \in D_{3 \times 3}.$$

Therefore $\lambda_1 X = x_{13}E_{13} + x_{12}E_{12} - x_{21}E_{21} - x_{31}E_{31}$

$$\Rightarrow \lambda_2(\lambda_1 X) = -x_{12}E_{12} - x_{21}E_{21}$$

$$\Rightarrow \lambda_3(\lambda_2(\lambda_1 X)) = 0.$$

5. DERIVATIONS SATISFYING A POLYNOMIAL IDENTITY IN MORE THAN TWO VARIABLES

In this chapter we assume R is a prime ring, characteristic $R \neq 2$, and $\lambda, \delta, \gamma, \varepsilon$, and σ are derivations of R. From Posner's Proposition (2.1) we know $\lambda\delta = \gamma$ implies either $\lambda = 0$ or $\delta = 0$. What happens if $\lambda\delta = \gamma^2$? More generally, what happens if $\lambda\delta = \gamma\varepsilon$? Answers to these questions are given in Theorems 5.4 and 5.5. We also investigate the case where $\lambda\delta = \gamma^3$, and show in Theorem 5.7 that $\lambda\delta = \gamma^3$ implies either $\lambda = 0$ or $\delta = 0$. We begin with a few lemmas.

<u>Lemma 5.1</u> Assume f and g are functions of R into R. Then $f(x)zg(y) + g(x)zf(y) = 0$, $\forall\, x,z,y \in R$, implies either $f = 0$ or $g = 0$.

<u>Proof</u>. We have $f(x)zg(x) + g(x)zf(x) = 0$, $\forall\, x,z \in R$. Assume $f \neq 0$. Then $f(x_1) \neq 0$ for some $x_1 \in R$ implies $g(x_1) = 0$ by Proposition 2.7. Therefore $f(x_1)zg(y) = 0$, $\forall\, z,y \in R$, and since R is prime, $g = 0$. Similarly $g \neq 0$ implies $f = 0$.

<u>Lemma 5.2</u> Assume f and g are nonzero functions of R into R. Then $f(x)zg(y) - g(x)zf(y) = 0$, $\forall\, x,z,y \in R$, implies $f = cg$ for some $c \in C$.

<u>Proof</u>. If $\exists\, y \in R$ such that $g(y) \neq 0$ and $f(y) \neq 0$, then by Lemma 3.8 we are done. Assume no such y exists. Thus $\forall\, y \in R$, $g(y) = 0$ or $f(y) = 0$. However, $g \neq 0$ implies $\exists\, w \in R$ such that $g(w) \neq 0$ and therefore $f(w) = 0$. Then $f(x)zg(w) - g(x)zf(w) = f(x)zg(w) = 0$, $\forall\, x,z \in R$. By the primeness of R, $f = 0$, a contradiction.

<u>Lemma 5.3</u> If $\lambda\delta - \gamma\lambda = \sigma$, then either $\lambda = 0$ or $\delta = \gamma$.

<u>Proof.</u> Note that $\lambda(\delta-\gamma) = \lambda\delta - \lambda\gamma = \sigma + \gamma\lambda - \lambda\gamma = \sigma + [\gamma,\lambda]$. Since the commutator of two derivations is again a derivation, we use Proposition 2.1 to conclude that either $\lambda = 0$ or $\delta - \gamma = 0$.

<u>Theorem 5.4</u> If λ, δ, and γ are nonzero and $\lambda\delta - \gamma^2 = \sigma$, then $\lambda = c\gamma$ and $\delta = c^{-1}\gamma$ for some $c \in C$.

<u>Proof.</u> $\forall\ x,y \in R,\ (\lambda\delta - \gamma^2)(xy) = \sigma(xy)$

$$\Rightarrow \lambda\delta xy + \lambda x\delta y + \delta x\lambda y + x\lambda\delta y - \gamma^2 xz - 2\gamma x\gamma y - x\gamma^2 y = \sigma xy + x\sigma y$$

$$\Rightarrow \quad 2\gamma x\gamma y - \lambda x\delta y - \delta x\lambda y = 0. \tag{5.1}$$

Replacing x by xz yields

$$2\gamma xz\gamma y - \lambda xz\delta y - \delta xz\lambda y = 0. \tag{5.2}$$

We now fix $y \in R$ and use (5.2) to show $\lambda y = 0 \Leftrightarrow \delta y = 0 \Leftrightarrow \gamma y = 0$. Note that

$$\lambda y = 0 \text{ or } \delta y = 0 \Rightarrow 2\gamma yz\gamma y = 0,\ \forall\ z \in R \Rightarrow \gamma y = 0 \tag{5.3}$$

Also note that

$$\begin{aligned} \gamma y = 0 \Rightarrow\ & \lambda xz\delta y + \delta xz\lambda y = 0 \quad \forall\ x,z \in R \\ \Rightarrow\ & \lambda yz\delta y + \delta yz\lambda y = 0 \quad \forall\ z \in R \\ \Rightarrow\ & \text{either } \lambda y = 0 \text{ or } \delta y = 0 \text{ by Proposition 2.7.} \end{aligned}$$

However,

$$\lambda y \neq 0 \text{ and } \delta y = 0 \Rightarrow \delta xz\lambda y = 0 \quad \forall\ x,z \in R \Rightarrow \delta = 0, \text{ a contradiction,}$$

and

$$\lambda y = 0 \text{ and } \delta y \neq 0 \Rightarrow \lambda xz\delta y = 0 \quad \forall\ x,z \in R \Rightarrow \lambda = 0, \text{ a contradiction.}$$

Thus, $\gamma y = 0 \Rightarrow \lambda y = 0$ and $\delta y = 0$. (5.4)

Together (5.3) and (5.4) imply $\lambda y = 0 \Leftrightarrow \delta y = 0 \Leftrightarrow \gamma y = 0$. Therefore, $\exists\ y \in R$ such that $2\gamma y \neq 0$, $\delta y \neq 0$, and $\lambda y \neq 0$. Applying Lemma 3.8 to (5.2) we obtain $\gamma = a\lambda + b\delta$ for some $a,b \in C$. Replacing z by γz in (5.2) yields, $\forall\ x,z,y \in R$,

$$2\gamma x\gamma z\gamma y - \lambda x\gamma z\delta y - \delta x\gamma z\lambda y = 0.$$

Using (5.1) we get

$$\lambda x\delta z\gamma y + \delta x\lambda z\gamma y - \lambda x\gamma z\delta y - \delta x\gamma z\lambda y = 0.$$

Replacing x by xs and y by ty yields, $\forall\ x,s,z,t,y \in R$,

$$\lambda xs\delta zt\gamma y + \delta xs\lambda zt\gamma y - \lambda xs\gamma zt\delta y - \delta xs\gamma zt\lambda y = 0$$

$$\Rightarrow \lambda xs(\delta zt\gamma y - \gamma zt\delta y) + \delta xs(\lambda zt\gamma y - \gamma zt\lambda y) = 0. \qquad (5.5)$$

Case 1: Assume $\exists\ z,t,y \in R$ such that $(\delta zt\gamma y - \gamma zt\delta y) \neq 0$ and $(\lambda zt\gamma y - \gamma zt\lambda y) \neq 0$. By Lemma 3.8, $\lambda = d\delta$ for some $d \in C$. Therefore, $\gamma = a\lambda + b\delta = a(d\delta) + b\delta = (ad + b)\delta$ and letting $c = ad + b$ we get $\gamma = c\delta$. Using (5.2) we get

$$2c\delta xzc\delta x - d\delta xz\delta x - \delta xzd\delta x = 0$$

$$\Rightarrow (2c^2 - 2d)\delta xz\delta x = 0$$

$$\Rightarrow c^2 = d \text{ or } cd^{-1} = c^{-1}.$$

Therefore, $\gamma = c\delta = (cd^{-1})\lambda = c^{-1}\lambda$, as we desired.

Case 2: Assume $\nexists\ z,t,y \in R$ such that $(\delta zt\gamma y - \gamma zt\delta y) \neq 0$ and $(\lambda zt\gamma y - \gamma zt\lambda y) \neq 0$. This implies for each fixed $z,t,y \in R$, $\delta zt\gamma y - \gamma zt\delta y = 0$ or $\lambda zt\gamma y - \gamma zt\lambda y = 0$. If $\exists$ some $z,t,y \in R$, such that $\delta zt\gamma y - \gamma zt\delta y = 0$ $(\neq 0)$ and $\lambda zt\gamma y - \gamma zt\lambda y \neq 0$ $(= 0)$, then using (5.5)

we get $\delta = 0$, a contradiction ($\lambda = 0$, a contradiction). Therefore $\delta zt\gamma y - \gamma zt\delta y = 0$, $\forall\, z,t,y \in R$, or $\lambda zt\gamma y - \gamma zt\lambda y = 0$, $\forall\, z,t,y \in R$.

Case 2.1: If $\delta zt\gamma y - \gamma zt\delta y = 0$, $\forall\, z,t,y \in R$, then Lemma 5.2 implies $\gamma = c\delta$ for some $c \in C$. Using $\gamma = c\delta$ in (5.5) we get

$$\delta xs(\lambda ztc\delta y - c\delta zt\lambda y) = 0$$

$$\Rightarrow \lambda zt\delta y - \delta zt\lambda y = 0$$

$$\Rightarrow \lambda = d\delta \text{ for some } d \in C \text{ by Lemma 5.2.}$$

The proof is completed as in Case 1.

Case 2.2: If $\lambda zt\gamma y - \gamma zt\lambda y = 0$, $\forall\, z,t,y \in R$, then by Lemma 5.2, $\gamma = c^{-1}\lambda$ for some $c^{-1} \in C$. Using $\gamma = c^{-1}\lambda$ in (5.5) we get

$$\lambda xs(\delta ztc^{-1}\lambda y - c^{-1}\lambda zt\delta y) = 0$$

$$\Rightarrow \delta zt\lambda y - \lambda zt\delta y = 0$$

$$\Rightarrow \lambda = d\delta \text{ for some } d \in C \text{ by Lemma 5.2.}$$

Again the proof is completed as in Case 1.

Theorem 5.5 If λ, δ, γ, and ε are nonzero and $\lambda\delta - \gamma\varepsilon = \sigma$, then $\exists$ $c \in C$ such that

1) $\lambda = c\gamma$ and $\delta = c^{-1}\varepsilon$

or 2) $\lambda = c\varepsilon$ and $\delta = c^{-1}\gamma$.

Proof. $\forall\ x,y \in R,\ (\lambda\delta - \gamma\varepsilon)(xy) = \sigma(xy)$

$$\Rightarrow \lambda\delta xy + \lambda x\delta y + \delta x\lambda y + x\lambda\delta y - \gamma\varepsilon xy - \gamma x\varepsilon y - \varepsilon x\gamma y - x\gamma\varepsilon y = \sigma xy + x\sigma y$$

$$\Rightarrow \quad \lambda x\delta y + \delta x\lambda y - \gamma x\varepsilon y - \varepsilon x\gamma y = 0. \tag{5.6}$$

Replacing x by xz yields

$$\lambda xz\delta y + \delta xz\lambda y - \gamma xz\varepsilon y - \varepsilon xz\gamma y = 0. \tag{5.7}$$

Replacing z by λz yields

$$\lambda x\lambda z\delta y + \delta x\lambda z\lambda y - \gamma x\lambda z\varepsilon y - \varepsilon x\lambda z\gamma y = 0.$$

Using (5.6) we get

$$\lambda x\varepsilon z\gamma y + \lambda x\gamma z\varepsilon y - \lambda x\delta z\lambda y + \delta x\lambda z\lambda y - \gamma x\lambda z\varepsilon y - \varepsilon x\lambda z\gamma y = 0.$$

Replacing x by xs and y by ty yields

$$\lambda xs\varepsilon zt\gamma y + \lambda xs\gamma zt\varepsilon y - \lambda xs\delta zt\lambda y$$
$$+ \delta xs\lambda zt\lambda y - \gamma xs\lambda zt\varepsilon y - \varepsilon xs\lambda zt\gamma y = 0$$

$$\Rightarrow (\delta xs\lambda z - \lambda xs\delta z)t\lambda y + (\lambda xs\gamma z - \gamma xs\lambda z)t\varepsilon y$$
$$+ \quad (\lambda xs\varepsilon z - \varepsilon xs\lambda z)t\gamma y = 0. \tag{5.8}$$

Case 1: Assume $\nexists\ x,s,z \in R$ such that $\delta xs\lambda z - \lambda xs\delta z \neq 0$, $\lambda xs\gamma z - \gamma xs\lambda z \neq 0$, and $\lambda xs\varepsilon z - \varepsilon xs\lambda z \neq 0$. This implies for each fixed $x,s,z \in R$, $q_1 = \delta xs\lambda z - \lambda xs\delta z = 0$, $q_2 = \lambda xs\gamma z - \gamma xs\lambda z = 0$, or $q_3 = \lambda xs\varepsilon z - \varepsilon xs\lambda z = 0$. We now use (5.8) and investigate the following possibilities:

Case 1.1: If $\exists\ x,s,z \in R$ such that $q_1 \neq 0$ and $q_2 = q_3 = 0$, then $\lambda = 0$, a contradiction.

Case 1.2: If $\exists\ x,s,z \in R$ such that $q_2 \neq 0$ and $q_1 = q_3 = 0$, then $\varepsilon = 0$, a contradiction.

Case 1.3: If $\exists$ $x,s,z \in R$ such that $q_3 \neq 0$ and $q_1 = q_2 = 0$, then

$\gamma = 0$, a contradiction.

Case 1.4: If $\exists$ $x,s,z \in R$ such that $q_1 \neq 0$, $q_2 \neq 0$, and $q_3 = 0$, then $\lambda = c\varepsilon$ for some $c \in C$ by the left-right symmetry of Lemma 3.8. Therefore $\lambda\delta - \gamma\varepsilon = \sigma$ implies $\varepsilon\delta - (c^{-1}\gamma)\varepsilon = c^{-1}\sigma$ and we conclude $\delta = c^{-1}\gamma$ by Lemma 5.3.

Case 1.5: If $\exists$ $x,s,z \in R$ such that $q_1 \neq 0$, $q_3 \neq 0$, and $q_2 = 0$, then $\lambda = c\gamma$ for some $c \in C$ by Lemma 3.8. Using $\lambda = c\gamma$ in (5.7) we get

$$c\gamma xz\delta y + \delta xzc\gamma y - \gamma xz\varepsilon y - \varepsilon xz\gamma y = 0$$

$$\Rightarrow \gamma xz(c\,\delta - \varepsilon)y + (c\,\delta - \varepsilon)xz\gamma y = 0$$

$$\Rightarrow c\,\delta - \varepsilon = 0 \text{ or } \delta = c^{-1}\varepsilon \text{ by Lemma (5.1).}$$

Case 1.6: If $\exists$ $x,s,z \in R$ such that $q_2 \neq 0$, $q_3 \neq 0$, and $q_1 = 0$, then $\gamma = b\varepsilon$ for some $b \in C$ by Lemma 3.8. Therefore $\lambda\delta - b\varepsilon^2 = \sigma$ implies $(b^{-1}\lambda)\,\delta - \varepsilon^2 = b^{-1}\sigma$. By Theorem 5.4, $b^{-1}\lambda = c\varepsilon$ and $\delta = c^{-1}\varepsilon$ for some $c \in C$. Note that $b^{-1}\lambda = c\varepsilon$ yields $\lambda = bc\varepsilon = c\gamma$.

Case 1.7: If $q_1 = \delta xs\lambda z - \lambda xs\delta z = 0$, $\forall$ $x,s,z \in R$, then $\lambda = b\delta$ for some $b \in C$ by Lemma 5.2. Therefore $b\delta^2 - \gamma\varepsilon = \sigma$ implies $(b^{-1}\gamma)\varepsilon - \delta^2 = -b^{-1}\sigma$. By Theorem 5.4, $b^{-1}\gamma = c^{-1}\delta$ and $\varepsilon = c\,\delta$ for some $c \in C$. Notice that $\lambda = b\delta = b(cb^{-1}\gamma) = c\gamma$.

Case 1.8: If $q_2 = \lambda xs\gamma z - \gamma xs\lambda z = 0$, $\forall$ $x,s,z \in R$, then $\lambda = c\gamma$, for some $c \in C$ by Lemma 5.2. We have as in Case 1.5, $\delta = c^{-1}\varepsilon$.

Case 1.9: If $q_3 = \lambda xs\varepsilon z - \varepsilon xs\lambda z = 0$, $\forall\, x,s,z \in R$, then $\lambda = c\varepsilon$ for some $c \in C$ by Lemma 5.2. We have as in Case 1.4, $\delta = c^{-1}\gamma$.

Case 2: Assume $\exists\, x,s,z \in R$ such that $\delta xs\lambda z - \lambda xs\delta z \neq 0$, $\lambda xs\gamma z - \gamma xs\lambda z \neq 0$, and $\lambda xs\varepsilon z - \varepsilon xs\lambda z \neq 0$. By Lemma 3.8, $\gamma = a\lambda + d\varepsilon$ for some $a,d \in C$. If $a = 0$, then $\gamma = d\varepsilon$ (Case 1.6). If $d = 0$ then $\gamma = a\lambda$ (Case 1.5). Therefore assume $a \neq 0$ and $d \neq 0$. Using (5.8) we get

$$(\delta xs\lambda z - \lambda xs\delta z)t\lambda y + (a\lambda xs\lambda z + d\lambda xs\varepsilon z - a\lambda xs\lambda z - d\varepsilon xs\lambda z)t\varepsilon y$$

$$+ a(\lambda xs\varepsilon z - \varepsilon xs\lambda z)t\lambda y + d(\lambda xs\varepsilon z - \varepsilon xs\lambda z)t\varepsilon y = 0$$

$$\Rightarrow [(\delta xs\lambda z - \lambda xs\delta z) + a(\lambda xs\varepsilon z - \varepsilon xs\lambda z)]t\lambda y$$

$$+ 2d(\lambda xs\varepsilon z - \varepsilon xs\lambda z)t\varepsilon y = 0. \tag{5.9}$$

Case 2.1: Assume $\exists\, x,s,z \in R$ such that

$$p_1 = [(\delta xs\lambda z - \lambda xs\delta z) + a(\lambda xs\varepsilon z - \varepsilon xs\lambda z)] \neq 0$$

and

$$p_2 = 2d(\lambda xs\varepsilon z - \varepsilon xs\lambda z) \neq 0.$$

Then $\lambda = c\,\varepsilon$ for some $c \in C$ by Lemma 3.8 and as in Case 1.4, $\delta = c^{-1}\gamma$.

Case 2.2: Assume $\nexists\, x,s,z \in R$ such that $p_1 \neq 0$ and $p_2 \neq 0$. Then for each fixed $x,s,z \in R$, either $p_1 = 0$ or $p_2 = 0$. If $\exists\, x,s,z \in R$ such that $p_1 = 0$ ($\neq 0$) and $p_2 \neq 0$ ($= 0$), then $\varepsilon = 0$, a contradiction ($\lambda = 0$, a contradiction). We conclude therefore that $\forall\, x,s,z \in R$,

$$(\delta xs\lambda z - \lambda xs\delta z) + a(\lambda xs\varepsilon z - \varepsilon xs\lambda z) = 0$$

$$\Rightarrow \quad (\delta - a\varepsilon)xs\lambda z - \lambda xs(\delta - a\varepsilon)z = 0. \tag{5.10}$$

If $\delta = a\varepsilon$, then (5.9) becomes

$$[a(\varepsilon xs\lambda z - \lambda xs\varepsilon z) + a(\lambda xs\varepsilon z - \varepsilon xs\lambda z)]t\lambda y$$

$$+ 2d(\lambda xs\varepsilon z - \varepsilon xs\lambda z)t\varepsilon y = 0$$

$$\Rightarrow \quad 2d(\lambda xs\varepsilon z - \varepsilon xs\lambda z)t\varepsilon y = 0$$

$$\Rightarrow \quad \lambda xs\varepsilon z - \varepsilon xs\lambda z = 0, \text{ a contradiction.}$$

Therefore $\delta \neq a\varepsilon$ and we may apply Lemma 5.2 to (5.10) to conclude that $\delta - a\varepsilon = e\lambda$ for some $e \in C$, $e \neq 0$. We now have $\gamma = a\lambda + d\varepsilon$ and $\delta = e\lambda + a\varepsilon$. Using (5.7) we obtain

$$\lambda xze\lambda y + \lambda xza\varepsilon y + e\lambda xz\lambda y + a\varepsilon xz\lambda y - a\lambda xz\varepsilon y - d\varepsilon xz\varepsilon y$$

$$- \varepsilon xza\lambda y - \varepsilon xzd\varepsilon y = 0$$

$$\Rightarrow \quad 2e\lambda xz\lambda y - 2d\varepsilon xz\varepsilon y = 0$$

$$\Rightarrow \quad e\lambda xz\lambda y - d\varepsilon xz\varepsilon y = 0.$$

Then $\exists$ $y \in R$ such that $e\lambda y \neq 0$ and $d\varepsilon y \neq 0$. (If not then $\lambda = 0$ or $\varepsilon = 0$, a contradiction). Therefore $\lambda = c\varepsilon$ for some $c \in C$ by Lemma 3.8 and as in Case 1.4 we have $\delta = c^{-1}\gamma$.

<u>Lemma 5.6</u> If $c\lambda^2 + \delta^3 = \sigma$ where $c \in C$, $c \neq 0$, then $\lambda = 0$.

<u>Proof</u>. $\forall$ $x,y \in R$, $(c\lambda^2 + \delta^3)(xy) = \sigma(xy)$

$$\Rightarrow c\lambda^2 xy + 2c\lambda x\lambda y + cx\lambda^2 y + \delta^3 xy + 3\delta^2 x\delta y + 3\delta x\delta^2 y + x\delta^3 y = \sigma xy + x\sigma y$$

$$\Rightarrow \quad 2c\lambda x\lambda y + 3\delta^2 x\delta y + 3\delta x\delta^2 y = 0. \tag{5.11}$$

Replacing x by xz yields

$$2c\lambda xz\lambda y + 2cx\lambda z\lambda y + 3\delta^2 xz\delta y + 6\delta x\delta z\delta y + 3x\delta^2 z\delta y$$

$$+ \quad 3\delta xz\delta^2 y + 3x\delta z\delta^2 y = 0$$

$$\Rightarrow\; 2c\lambda xz\lambda y + 3\delta^2 xz\delta y + 6\delta x\delta z\delta y + 3\delta xz\delta^2 y = 0.$$

Replacing z by δz yields

$$2c\lambda x\delta z\lambda y + 3\delta^2 x\delta z\delta y + 6\delta x\delta^2 z\delta y + 3\delta x\delta z\delta^2 y = 0.$$

Using (5.11) we get

$$2c\lambda x\delta z\lambda y - 2c\lambda x\lambda z\delta y - 2c\delta x\lambda z\lambda y = 0$$

$$\Rightarrow\; \lambda x\delta z\lambda y - \lambda x\lambda z\delta y - \delta x\lambda z\lambda y = 0.$$

Replacing x by xs and y by yt yields

$$\lambda xs\delta zy\lambda t - \lambda xs\lambda zy\delta t - \delta xs\lambda zy\lambda t = 0$$

$$\Rightarrow\; \lambda xs(\delta xy\lambda t - \lambda xy\delta t) - \delta xs(\lambda xy\lambda t) = 0. \qquad (5.12)$$

Case 1: Assume $\exists$ $z,y,t \in R$ such that $\delta zy\lambda t - \lambda zy\delta t \neq 0$ and $\lambda xy\lambda t \neq 0$. Then $\delta = a\lambda$ for some $a \in C$ by Lemma 3.8. Using (5.12) we get

$$\lambda xs(a\lambda zy\lambda t - \lambda zya\lambda t) - a\lambda xs(\lambda zy\lambda t) = 0$$

$$\Rightarrow\; \lambda xs\lambda zy\lambda t = 0$$

$$\Rightarrow\; \lambda = 0.$$

Case 2: Assume $\nexists$ $z,y,t \in R$ such that $\delta zy\lambda t - \lambda zy\delta t \neq 0$ and $\lambda zy\lambda t \neq 0$. This implies for each fixed $z,y,t \in R$, either $\delta zy\lambda t - \lambda zy\delta t = 0$ or $\lambda xy\lambda t = 0$. If $\exists$ $x,y,t \in R$ such that $\delta zy\lambda t - \lambda zy\delta t = 0$ ($\neq 0$) and $\lambda zy\lambda t \neq 0$ ($= 0$), then using (5.12) we get $\delta = 0$ ($\lambda = 0$). Notice that $\delta = 0$ implies $\lambda = 0$. Therefore assume $\lambda zy\lambda t = 0$, $\forall$ $z,y,t \in R$, or $\delta xy\lambda t - \lambda zy\delta t = 0$, $\forall$ $z,y,t \in R$. If

$\lambda xy\lambda t = 0$, then obviously $\lambda = 0$. If $\delta xy\lambda t - \lambda xy\delta t = 0$, then from (5.12) we obtain $\delta xs(\lambda zy\lambda t) = 0$. This implies $\delta = 0$ or $\lambda = 0$ and again we must have $\lambda = 0$.

<u>Theorem 5.7</u> If $\lambda\delta - \gamma^3 = \sigma$, then either $\lambda = 0$ or $\delta = 0$.

<u>Proof</u>. $\forall\ x,y \in R$, $(\lambda\delta - \gamma^3)(xy) = \sigma(xy)$

$$\Rightarrow \lambda\delta xy + \lambda x\delta y + \delta x\lambda y + x\lambda\delta y - \gamma^3 xy - 3\gamma^2 x\gamma y - 3\gamma x\gamma^2 y - x\gamma^3 y$$

$$= \sigma xy + x\sigma y$$

$$\Rightarrow \lambda x\delta y + \delta x\lambda y - 3\gamma^2 x\gamma y - 3\gamma x\gamma^2 y = 0. \tag{5.13}$$

Replacing x by xz yields

$$\lambda xz\delta y + x\lambda z\delta y + \delta xz\lambda y + x\delta z\lambda y - 3\gamma^2 xz\gamma y - 6\gamma x\gamma z\gamma y$$
$$- 3x\gamma^2 z\gamma y - 3\gamma xz\gamma^2 y - 3x\gamma z\gamma^2 y = 0$$

$$\Rightarrow \lambda xz\delta y + \delta xz\lambda y - 3\gamma^2 xz\gamma y - 6\gamma x\gamma z\gamma y - 3\gamma xz\gamma^2 y = 0.$$

Replacing z by γz yields

$$\lambda x\gamma z\delta y + \delta x\gamma z\lambda y - 3\gamma^2 x\gamma z\gamma y - 6\gamma x\gamma^2 z\gamma y - 3\gamma x\gamma z\gamma^2 y = 0.$$

Using (5.13) we get

$$\lambda x\gamma z\delta y + \delta x\gamma z\lambda y - \lambda x\delta z\gamma y - \delta x\lambda z\gamma y - \gamma x\lambda z\delta y - \gamma x\delta z\lambda y = 0.$$

Replacing x by xs and y by ty yields

$$\lambda xs\gamma zt\delta y + \delta xs\gamma zt\lambda y - \lambda xs\delta zt\gamma y - \delta xs\lambda zt\gamma y$$
$$- \gamma xs\lambda zt\delta y - \gamma xs\delta zt\lambda y = 0.$$

$$\Rightarrow \lambda xs(\gamma zt\delta y - \delta zt\gamma y) + \delta xs(\gamma zt\lambda y - \lambda zt\gamma y)$$
$$- \gamma xs(\lambda zt\delta y + \delta zt\lambda y) = 0. \tag{5.14}$$

<u>Case 1</u>: Assume $\nexists$ $z,t,y \in R$ such that $\gamma zt\delta y - \delta zt\gamma y \neq 0$, $\gamma zt\lambda y - \lambda zt\gamma y \neq 0$, and $\lambda zt\delta y + \delta zt\lambda y \neq 0$. This implies that for each fixed $z,t,y \in R$, $q_1 = \gamma zt\delta y = 0$, $q_2 = \gamma zt\lambda y - \lambda zt\gamma y = 0$, or $q_3 = \lambda zt\delta y - \delta zt\lambda y = 0$. Now use (5.14) and consider the following:

<u>Case 1.1</u>: If $\exists$ $z,t,y \in R$ such that $q_1 \neq 0$ and $q_2 = q_3 = 0$, then $\lambda = 0$.

<u>Case 1.2</u>: If $\exists$ $z,t,y \in R$ such that $q_2 \neq 0$ and $q_1 = q_3 = 0$, then $\delta = 0$.

<u>Case 1.3</u>: If $\exists$ $z,t,y \in R$ such that $q_3 \neq 0$ and $q_1 = q_2 = 0$, then $\gamma = 0$.

Notice that $\gamma = 0$ implies $\lambda\delta = \sigma$ and by Posner's result, either $\lambda = 0$ or $\delta = 0$.

<u>Case 1.4</u>: If $\exists$ $z,t,y \in R$ such that $q_1 \neq 0$, $q_2 \neq 0$, and $q_3 = 0$, then $\lambda = c\delta$ for some $c \in C$ by Lemma 3.8. Applying Lemma 5.6 to $c\delta^2 - \gamma^3 = \sigma$ we get $\delta = 0$.

<u>Case 1.5</u>: If $\exists$ $z,t,y \in R$ such that $q_1 \neq 0$, $q_2 = 0$, and $q_3 \neq 0$, then $\gamma = c\lambda$ for some $c \in C$ by Lemma 3.8. Use (5.14) to obtain $\forall$ $x,s,z,t,y \in R$,

$$\lambda xs(c\lambda zt\delta y - \delta ztc\lambda y) - c\lambda xs(\lambda zt\delta y + \delta zt\lambda y) = 0$$

$$\Rightarrow \quad c\lambda xs(\lambda zt\delta y - \delta zt\lambda y - \lambda zt\delta y - \delta zt\lambda y) = 0$$

$$\Rightarrow \quad \lambda xs\delta zt\lambda y = 0$$

$$\Rightarrow \quad \text{either } \lambda = 0 \text{ or } \delta = 0.$$

<u>Case 1.6</u>: If $\exists$ $z,t,y \in R$ such that $q_1 = 0$, $q_2 \neq 0$, and $q_3 \neq 0$, then $\gamma = c\delta$ for some $c \in C$ by Lemma 3.8. Use (5.14) to obtain $\forall$ $x,s,z,t,y \in R$,

$$\delta xs(c\delta zt\lambda y - \lambda ztc\delta y) - c\delta xs(\lambda zt\delta y + \delta zt\lambda y) = 0$$

$$\Rightarrow \quad c\delta xs(\delta zt\lambda y - \lambda zt\delta y - \lambda zt\delta y - \delta zt\lambda y) = 0$$

$$\Rightarrow \quad \delta x s \lambda z t \delta y = 0$$

$$\Rightarrow \quad \text{either } \delta = 0 \text{ or } \lambda = 0.$$

<u>Case 1.7</u>: If $q_1 = \gamma z t \delta y - \delta z t \gamma y = 0$, $\forall\, z,t,y \in R$, then $\gamma = 0$, $\delta = 0$, or $\delta = c\gamma$ for some $c \in C$ by Lemma 5.2. This is Case 1.6.

<u>Case 1.8</u>: If $q_2 = \gamma z t \lambda y - \lambda z t \gamma y = 0$, $\forall\, z,t,y \in R$, then $\gamma = 0$, $\lambda = 0$, or $\gamma = c\lambda$ for some $c \in C$ by Lemma 5.2. This is Case 1.5.

<u>Case 1.9</u>: If $q_3 = \lambda z t \delta y + \delta z t \lambda y = 0$, $\forall\, z,t,y \in R$, then by Lemma 5.1, either $\lambda = 0$ or $\delta = 0$.

<u>Case 2</u>: Assume $\exists\, z,t,y \in R$ such that $q_1 \neq 0$, $q_2 \neq 0$, and $q_3 \neq 0$. By Lemma 3.8, $\lambda = c\delta + d\gamma$ for some $c,d \in C$, $c \neq 0$ or $d \neq 0$. If $c = 0$, then $\lambda = d\gamma$ and we have Case 1.5. If $d = 0$, then $\lambda = c\delta$ and we have Case 1.4. If $c \neq 0$ and $d \neq 0$, then substitute $c\delta + d\gamma$ for λ in (5.14) to get $\forall\, x,s,z,t,y \in R$,

$$\begin{aligned} &c\delta xs(\gamma zt\delta y - \delta zt\gamma y) + d\gamma xs(\gamma zt\delta y - \delta zt\gamma y) \\ &+ \delta xs(\gamma ztc\delta y + \gamma ztd\gamma y - c\delta zt\gamma y - d\gamma zt\gamma y) \\ &- \gamma xs(c\delta zt\delta y + d\gamma zt\delta y + \delta ztc\delta y + \delta ztd\gamma y) = 0 \end{aligned}$$

$$\Rightarrow \quad 2\delta xs(c\gamma zt\delta y - c\delta zt\gamma y) - 2\gamma xs(c\delta zt\delta y + d\delta zt\gamma y) = 0$$

$$\Rightarrow \quad \delta xs(c\gamma zt\delta y - c\delta zt\gamma y) - \gamma xs(c\delta zt\delta y + d\delta zt\gamma y) = 0. \qquad (5.15)$$

<u>Case 2.1</u>: Assume $\nexists\, z,t,y \in R$ such that $c\gamma zt\delta y - c\delta zt\gamma y \neq 0$, and $c\delta zt\delta y + d\delta zt\gamma y \neq 0$. Then for each fixed $z,t,y \in R$, either $p_1 = c\gamma zt\delta y - c\delta zt\gamma y = 0$ or $p_2 = c\delta zt\delta y + d\delta zt\gamma y = 0$. If $\exists\, z,t,y \in R$ such that $p_1 = 0$ ($\neq 0$) and $p_2 \neq 0$ ($= 0$), then $\gamma = 0$ ($\delta = 0$). Therefore assume $c\delta zt\delta y + d\delta zt\gamma y = 0$, $\forall\, z,t,y \in R$. Using this and (5.15) we get

$\delta xs(c\gamma xt\delta y - c\delta zt\gamma y)$, $\forall\, x,s,z,t,y \in R$. Since $\exists\, z,t,y \in R$ such that $q_1 \neq 0$, we conclude that $\delta = 0$.

<u>Case 2.2</u>: Assume $\exists\, z,t,y \in R$ such that $p_1 \neq 0$ and $p_2 \neq 0$. Then by applying Lemma 3.8 to (5.15) we obtain $\gamma = a\delta$ for some $a \in C$. This is just Case 1.6.

6. DERIVATIONS SATISFYING OTHER IDENTITIES

6.1 Identity $\lambda\delta\lambda = 0$

Assume λ and δ are derivations of a prime ring R. By Lemma 4.1, $\lambda\delta = 0$ implies $\lambda = 0$ or $\delta^2 = 0$. What happens if $\lambda\delta\lambda = 0$? A simple example indicates that we cannot conclude that either λ or δ is nilpotent.

Example 1: Let R be the 2×2 matrix ring over a division ring and let λ and δ be defined by

$$\lambda\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right] = \begin{pmatrix} 0 & x_{12} \\ -x_{21} & 0 \end{pmatrix}$$

$$\delta\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right] = \begin{pmatrix} x_{21} - x_{12} & x_{22} - x_{11} \\ -(x_{22} - x_{11}) & -(x_{21} - x_{12}) \end{pmatrix}$$

It follows easily that neither λ nor δ is nilpotent. However,

$$\lambda\delta\lambda\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \lambda\delta\begin{pmatrix} 0 & x_{12} \\ -x_{21} & 0 \end{pmatrix} = \lambda\begin{pmatrix} -x_{21} - x_{12} & 0 \\ 0 & x_{21} + x_{12} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Example 2: Let R be the ring of real quaternions. If we define λ and δ by $\lambda x = [i,x]$ and $\delta x = [j,x]$, then $\lambda\delta\lambda = \delta\lambda\delta = 0$. However, λ and δ are not nilpotent since in R, any derivation which annihilates the center is inner and any nilpotent inner derivation is induced by a nilpotent element.

Although $\lambda\delta\lambda = 0$ does not imply the nilpotency of λ or δ, several interesting statements can still be made. We first need two detailed lemmas.

<u>Lemma 6.1</u> If $\lambda\delta\lambda = 0$, then either $\delta^2\lambda = 0$ or $\lambda^2(\lambda x\lambda y)\lambda(\lambda z\lambda w) = 0$, $\forall\ x,y,z,w \in R$.

<u>Proof.</u> For all $x,y \in R$, $\lambda\delta\lambda(xy) = 0$

$$\Rightarrow \delta\lambda x\lambda y + \lambda^2 x\delta y + \lambda x\lambda\delta y + \lambda\delta x\lambda y + \delta x\lambda^2 y + \lambda x\delta\lambda y = 0.$$

Replacing x by $\delta\lambda x$ yields $\lambda\delta^2 x\lambda y + \delta^2\lambda x\lambda^2 y = 0$ (6.1)

or

$$\lambda a\lambda y + a\lambda^2 y = 0 \quad \text{where} \quad a = \delta^2\lambda x.$$

Replacing y by yz yields $\lambda a(\lambda yz + y\lambda z) + a(\lambda^2 yz + 2\lambda y\lambda z + y\lambda^2 z) = 0$

or

$$\lambda a(y\lambda z) + a(2\lambda y\lambda z + y\lambda^2 z) = 0.$$

Replacing y by λy yields $\lambda a(\lambda y\lambda z) + a(2\lambda^2 y\lambda z + \lambda y\lambda^2 z) = 0$

or

$$a(\lambda^2 y\lambda z + \lambda y\lambda^2 z) = 0.$$

Replacing z by zw yields $a(\lambda^2 y\lambda zw + \lambda^2 yz\lambda w + \lambda y\lambda^2 zw + 2\lambda y\lambda z\lambda w$
$+ \lambda yz\lambda^2 w) = 0.$

Replacing z by λz yields $a(\lambda^2 y\lambda z\lambda w + 2\lambda y\lambda^2 z\lambda w + \lambda y\lambda z\lambda^2 w) = 0$

or

$$a\lambda y(\lambda^2 z\lambda w + \lambda z\lambda^2 w) = 0.$$

Replacing y by yv yields $a(\lambda yv + y\lambda v)\lambda(\lambda z\lambda w) = 0.$

Replacing v by $\lambda(\lambda s\lambda t)$ yields $a(\lambda y\lambda(\lambda s\lambda t) + y\lambda^2(\lambda s\lambda t))\lambda(\lambda z\lambda w) = 0$

or

$$ay\lambda^2(\lambda s\lambda t)\lambda(\lambda z\lambda w) = 0.$$

Thus, either $\delta^2\lambda x = 0$, $\forall\ x \in R$ or $\lambda^2(\lambda s\lambda t)\lambda(\lambda z\lambda w) = 0$, $\forall\ s,t,z,w \in R$.

<u>Lemma 6.2</u> If $\lambda^2(\lambda x\lambda y)\lambda(\lambda z\lambda w) = 0$, $\forall\ x,y,z,w \in R$, then either $\lambda^5 = 0$ or $\lambda(\lambda x\lambda y) = 0$, $\forall\ x,y \in R$.

<u>Proof.</u> Assuming $b = \lambda(\lambda x\lambda y)$ we have $\lambda b\lambda(\lambda z\lambda w) = 0$.

Replacing w by wv yields $\lambda b(\lambda^2 z\lambda wv + \lambda^2 zw\lambda v + \lambda z\lambda^2 wv + 2\lambda z\lambda w\lambda v$

$$+ \lambda zw\lambda^2 v) = 0.$$

Replacing w by λw yields $\lambda b(\lambda^2 z\lambda w\lambda v + 2\lambda z\lambda^2 w\lambda v + \lambda z\lambda w\lambda^2 v) = 0$

or

$$\lambda b\lambda z\lambda(\lambda w\lambda v) = 0.$$

Replacing z by zu yields $\lambda b(\lambda zu + z\lambda u)\lambda(\lambda w\lambda v) = 0.$ (6.2)

Replacing z by $\lambda z\lambda t$ in (6.2) yields $\lambda b\lambda z\lambda t\lambda u\lambda(\lambda w\lambda v) = 0$.

Replacing u by $\lambda u\lambda t$ in (6.2) yields $\lambda bz\lambda(\lambda u\lambda r)\lambda(\lambda w\lambda v) = 0$.

Thus either $\lambda^2(\lambda x\lambda y) = 0$ or $\lambda(\lambda u\lambda r)\lambda(\lambda w\lambda v) = 0$.

<u>Case 1</u>: $\lambda^2(\lambda x\lambda y) = 0 \Rightarrow \lambda^3 x\lambda y + 2\lambda^2 x\lambda^2 y + \lambda x\lambda^3 y = 0$.

Replacing x by xs yields $(\lambda^3 xs + 3\lambda^2 x\lambda s + 3\lambda x\lambda^2 s + x\lambda^3 s)\lambda y$

$$+\ 2(\lambda^2 xs + 2\lambda x\lambda s + x\lambda^2 s)\lambda^2 y + (\lambda xs + x\lambda s)\lambda^3 y = 0$$

or

$$\lambda^3 xs\lambda y + 3\lambda^2 x\lambda s\lambda y + 3\lambda x\lambda^2 s\lambda y + 2\lambda^2 xs\lambda^2 y$$

$$+ 4\lambda x\lambda s\lambda^2 y + \lambda xs\lambda^3 y = 0.$$

Replacing s by $\lambda^2 s$ yields $\lambda^2 x\lambda^3 s\lambda y + 2\lambda x\lambda^4 s\lambda y + 2\lambda^2 x\lambda^2 s\lambda^2 y$

$$+ 4\lambda x\lambda^3 s\lambda^2 y + \lambda x\lambda^2 s\lambda^3 y = 0$$

$$\Rightarrow \quad \lambda^2 x\lambda^3 s\lambda y + \lambda x\lambda^4 s\lambda y + 2\lambda^2 x\lambda^2 s\lambda^2 y + 2\lambda x\lambda^3 s\lambda^2 y = 0$$

$$\Rightarrow \quad \lambda^2 x\lambda^3 s\lambda y + 2\lambda^2 x\lambda^2 s\lambda^2 y - \lambda x\lambda^2 s\lambda^3 y = 0$$

$$\Rightarrow \quad -\lambda^2 x\lambda s\lambda^3 y - \lambda x\lambda^2 s\lambda^3 y = 0$$

$$\Rightarrow \quad \lambda(\lambda x\lambda s)\lambda^3 y = 0.$$

Using Lemma 3.14 we obtain that either $\lambda^5 = 0$ or $\lambda(\lambda x\lambda s) = 0$, $\forall\ x,s \in R$.

<u>Case 2</u>: $\lambda(\lambda u\lambda r)\lambda(\lambda w\lambda v) = 0 \Rightarrow \lambda(\lambda u\lambda r)(\lambda^2 w\lambda v + \lambda w\lambda^2 v) = 0.$ (6.3)

Replacing w by wp in (6.3) yields

$$\lambda(\lambda u\lambda r)(\lambda^2 wp\lambda v + 2\lambda w\lambda p\lambda v + w\lambda^2 p\lambda v + \lambda wp\lambda^2 v + w\lambda p\lambda^2 v) = 0.$$

Replacing w by $\lambda w\lambda q$ yields $\lambda(\lambda u\lambda r)\lambda w\lambda q\lambda(\lambda p\lambda v) = 0.$

Replacing v by vg in (6.3) yields

$$\lambda(\lambda u\lambda r)(\lambda^2 w\lambda vg + \lambda^2 wv\lambda g + \lambda w\lambda^2 vg + 2\lambda w\lambda v\lambda g + \lambda wv\lambda^2 g) = 0$$

or

$$\lambda(\lambda u\lambda r)(\lambda^2 wv\lambda g + 2\lambda w\lambda v\lambda g + \lambda wv\lambda^2 g) = 0.$$

Replacing v by λv yields $\lambda(\lambda u\lambda r)\lambda w\lambda(\lambda v\lambda g) = 0.$

Replacing w by $w\lambda h$ yields $\lambda(\lambda u\lambda r)(\lambda w\lambda h + w\lambda^2 h)\lambda(\lambda v\lambda g) = 0$

or

$$\lambda(\lambda u\lambda r)w\lambda^2 h\lambda(\lambda v\lambda g) = 0.$$

Thus, either $\lambda(\lambda u\lambda r) = 0$ or $\lambda^2 h\lambda(\lambda v\lambda g) = 0.$

As a final note, applying Lemma 3.14 to $\lambda^2 h\lambda(\lambda v\lambda g) = 0$ yields

$$\lambda^3 = 0 \quad \text{or} \quad \lambda(\lambda v\lambda g) = 0 \quad \forall\ v,g \in R.$$

<u>Theorem 6.3</u> If $\lambda\delta\lambda = 0$ and if neither λ nor δ is nilpotent, then λ^{2k+1} is a derivation, $\forall\ k \in Z^+$.

<u>Proof.</u> Combining Lemmas 6.1 and 6.2 we have either $\delta^2\lambda = 0$, $\lambda^5 = 0$, or $\lambda(\lambda x\lambda y) = 0$, $\forall\ x,y \in R$. Considering the hypotheses and Theorem 4.3, we must have $\lambda(\lambda x\lambda y) = 0$, $\forall\ x,y \in R$. The proof is complete by using Lemma 3.5.

<u>Theorem 6.4</u> If $\lambda\delta\lambda = 0$ and if neither λ nor δ is nilpotent, then $\lambda^2\delta^2 = \delta^2\lambda^2$.

<u>Proof.</u> $\lambda\delta\lambda(x\lambda y) = 0,\ \forall\ x,y \in R \Rightarrow \lambda\delta(\lambda x\lambda y + x\lambda^2 y) = 0$

$$\Rightarrow \lambda(\delta\lambda x\lambda y + \lambda x\delta\lambda y + \delta x\lambda^2 y + x\delta\lambda^2 y) = 0$$

$$\Rightarrow \delta\lambda x\lambda^2 y + \lambda^2 x\delta\lambda y + \lambda\delta x\lambda^2 y + \delta x\lambda^3 y + \lambda x\delta\lambda^2 y = 0. \tag{6.4}$$

$$\lambda^2\delta\lambda(xy) = 0,\ \forall\ x,y \in R \Rightarrow \lambda^2\delta(\lambda xy + x\lambda y) = 0$$

$$\Rightarrow \lambda^2(\delta\lambda xy + \lambda x\delta y + \delta x\lambda y + x\delta\lambda y) = 0$$

$$\Rightarrow \delta\lambda x\lambda^2 y + \lambda^3 x\delta y + \lambda^2 x\lambda\delta y + \lambda\delta x\lambda^2 y + \delta x\lambda^3 y + \lambda^2 x\delta\lambda y = 0. \tag{6.5}$$

Subtracting (6.4) from (6.5) gives $\lambda^3 x\delta y + \lambda^2 x\lambda\delta y - \lambda x\delta\lambda^2 y = 0$.

Replacing y by λy yields $\quad \lambda^3 x\delta\lambda y - \lambda x\delta\lambda^3 y = 0. \qquad (6.6)$

Replacing x by xz yields $\quad \lambda^3 xz\delta\lambda y + x\lambda^3 z\delta\lambda y - \lambda xz\delta\lambda^3 y - x\lambda z\delta\lambda^3 y = 0.$

Replacing z by $\delta\lambda z$ yields $\quad \lambda^3 x\delta\lambda z\delta\lambda y - \lambda x\delta\lambda z\delta\lambda^3 y = 0$

or

$$\lambda x\delta\lambda^3 z\delta\lambda y - \lambda x\delta\lambda z\delta\lambda^3 y = 0.$$

By Lemma 3.14 we have $\quad \delta\lambda^3 z\delta\lambda y - \delta\lambda z\delta\lambda^3 y = 0.$

From (6.6) we know $\quad \delta(\lambda^3 x\delta\lambda y - \lambda x\delta\lambda^3 y) = 0$

$$\Rightarrow \quad \delta\lambda^3 x\delta\lambda y + \lambda^3 x\delta^2\lambda y - \delta\lambda x\delta\lambda^3 y - \lambda x\delta^2\lambda^3 y = 0$$

$$\Rightarrow \quad \lambda^3 x\delta^2\lambda y - \lambda x\delta^2\lambda^3 y = 0.$$

Using (6.1) we have $\quad -\lambda^2 x\lambda\delta^2\lambda y - \lambda x\delta^2\lambda^3 y = 0.$

Using $\lambda(\lambda s\lambda t) = 0$, it follows that $\quad \lambda x\lambda^2\delta^2\lambda y - \lambda x\delta^2\lambda^3 y = 0.$

By Lemma 3.14 we have $\quad \lambda^2\delta^2\lambda - \delta^2\lambda^3 = 0$

or

$$(\lambda^2\delta^2 - \delta^2\lambda^2)\lambda = 0.$$

We know that $[[[\delta,\lambda],\lambda],\delta] = \lambda^2\delta^2 - \delta^2\lambda^2$ is a derivation. Therefore by Lemma 4.1, $\lambda^2\delta^2 - \delta^2\lambda^2 = 0.$

Theorem 6.5. If $\lambda\delta\lambda = 0$ and if neither λ nor δ is nilpotent, then $\lambda\delta^{2k+1}\lambda = 0, \ \forall\, k \in Z^+.$

Proof. Theorem 6.4 implies $(\lambda^2\delta^2 - \delta^2\lambda^2)\delta\lambda = \lambda^2\delta^3\lambda = 0.$ In Theorem 6.3 we saw that $\lambda(\lambda x\lambda y) = 0, \ \forall\, x,y \in R.$ Substituting $\delta^3\lambda x$ for x we get $\lambda(\lambda\delta^3\lambda x\lambda y) = \lambda^2\delta^3\lambda x\lambda y + \lambda\delta^3\lambda x\lambda^2 y = \lambda\delta^3\lambda x\lambda^2 y = 0.$ Using Lemma 3.14 and the fact that λ is not nilpotent we conclude

$\lambda\delta^3\lambda = 0$. Starting with $(\lambda^2\delta^2 - \delta^2\lambda^2)\delta^3\lambda = \lambda^2\delta^5\lambda = 0$, we repeat the same argument to obtain $\lambda\delta^5\lambda = 0$. We continue this process to arrive at $\lambda\delta^{2k+1}\lambda = 0, \ \forall\, k \in Z^+$.

6.2 Identity $\lambda\delta^2\lambda = 0$

Theorem 6.5 If $\lambda\delta^2\lambda = 0$, characteristic $R \neq 2$, and R has no zero divisors, then either λ or δ is nilpotent.

Proof. $\lambda\delta^2\lambda(xy) = 0, \ \forall\, x,y \in R, \Rightarrow \ \lambda\delta^2(\lambda xy + x\lambda y) = 0$

$$\Rightarrow \lambda[\delta^2\lambda xy + 2\delta\lambda x\delta y + \lambda x\delta^2 y + \delta^2 x\lambda y + 2\delta x\delta\lambda y + x\delta^2\lambda y] = 0$$

$$\Rightarrow \delta^2\lambda x\lambda y + 2\lambda\delta\lambda x\delta y + 2\delta\lambda x\lambda\delta y + \lambda^2 x\delta^2 y + \lambda x\lambda\delta^2 y$$
$$+ \lambda\delta^2 x\lambda y + \delta^2 x\lambda^2 y + 2\lambda\delta x\delta\lambda y + 2\delta x\lambda\delta\lambda y + \lambda x\delta^2\lambda y = 0.$$

Replacing x by $\delta^2\lambda x$ yields $\lambda\delta^4\lambda x\lambda y + \delta^4\lambda x\lambda^2 y + 2\lambda\delta^3\lambda x\delta\lambda y$

$$+ 2\delta^3\lambda x\lambda\delta\lambda y = 0.$$

Replacing y by yz yields $\lambda\delta^4\lambda x\lambda yz + \lambda\delta^4\lambda xy\lambda z + \delta^4\lambda x\lambda^2 yz$

$$+ 2\delta^4\lambda x\lambda y\lambda z + \delta^4\lambda xy\lambda^2 z + 2\lambda\delta^3\lambda x\delta\lambda yz + 2\lambda\delta^3\lambda x\lambda y\delta z$$
$$+ 2\lambda\delta^3\lambda x\delta y\lambda z + 2\lambda\delta^3\lambda xy\delta\lambda z + 2\delta^3\lambda x\lambda\delta\lambda yz + 2\delta^3\lambda x\delta\lambda y\lambda z$$
$$+ 2\delta^3\lambda x\lambda^2 y\delta z + 2\delta^3\lambda x\lambda y\lambda\delta z + 2\delta^3\lambda x\lambda\delta y\lambda z + 2\delta^3\lambda x\delta y\lambda^2 z$$
$$+ 2\delta^3\lambda x\lambda y\delta\lambda z + 2\delta^3\lambda xy\lambda\delta\lambda z = 0.$$

Replacing z by $\delta^3\lambda z$ yields $2\lambda\delta^3\lambda x\lambda y\delta^3\lambda z + 2\delta^3\lambda x\lambda^2 y\delta^3\lambda z$

$$+ 2\delta^3\lambda x\lambda y\lambda\delta^3\lambda z = 0$$

or

$$\lambda\delta^3\lambda x\lambda y\delta^3\lambda z + \delta^3\lambda x\lambda^2 y\delta^3\lambda z + \delta^3\lambda x\lambda y\lambda\delta^3\lambda z = 0.$$

Letting $\delta^3\lambda x = a$ we obtain $\lambda a\lambda ya + a\lambda^2 ya + a\lambda y\lambda a = 0,$ (6.7)

which by replacing y by yt yields

$$\lambda a\lambda yta + \lambda ay\lambda ta + a\lambda^2 yta + 2a\lambda y\lambda ta + ay\lambda^2 ta$$

$$+ \ a\lambda yt\lambda a + ay\lambda t\lambda a = 0.$$

Letting $y = a$ we obtain $\lambda a\lambda ata + \lambda aa\lambda ta + a\lambda^2 ata + a\lambda a\lambda ta$

$$+ \ a\lambda at\lambda a = 0$$

which by replacing t by ta yields

$$\lambda aat\lambda aa + a\lambda ata\lambda a = 0. \qquad (6.8)$$

Replacing t by $t\lambda aaw\lambda a$ in (6.8) yields

$$\lambda aa(t\lambda aaw\lambda a)\lambda aa + a\lambda a(t\lambda aaw\lambda a)a\lambda a = 0$$

$$\Rightarrow \quad -a\lambda ata\lambda aw\lambda a\lambda aa + a\lambda at\lambda aaw\lambda aa\lambda a = 0$$

$$\Rightarrow \quad a\lambda at[-a\lambda aw\lambda a\lambda aa + \lambda aaw\lambda aa\lambda a] = 0$$

$$\Rightarrow \quad a\lambda a = 0 \quad \text{or} \quad -a\lambda aw\lambda a\lambda aa + \lambda aaw\lambda aa\lambda a = 0$$

$$\Rightarrow \quad a\lambda a = 0 \quad \text{or} \quad -a\lambda aw\lambda a\lambda aa - a\lambda awa\lambda a\lambda a = 0$$

$$\Rightarrow \quad a\lambda a = 0 \quad \text{or} \quad a\lambda aw[\lambda a\lambda aa + a\lambda a\lambda a] = 0$$

$$\Rightarrow \quad a\lambda a = 0 \quad \text{or} \quad \lambda a\lambda aa + a\lambda a\lambda a = 0.$$

Replacing t by $t\lambda aawa$ in (6.8) yields

$$\lambda aa(t\lambda aawa)\lambda aa + a\lambda a(t\lambda aawa)a\lambda a = 0$$

$$\Rightarrow \quad -a\lambda ata\lambda awa\lambda aa + a\lambda at\lambda aawaa\lambda a = 0$$

$$\Rightarrow \quad a\lambda at[-a\lambda awa\lambda aa + \lambda aawaa\lambda a] = 0$$

$\Rightarrow$ $a\lambda a = 0$ or $-a\lambda awa\lambda aa + \lambda aawaa\lambda a = 0$

$\Rightarrow$ $a\lambda a = 0$ or $\lambda aaw\lambda aaa + \lambda aawaa\lambda a = 0$

$\Rightarrow$ $a\lambda a = 0$ or $\lambda aaw[\lambda aaa + aa\lambda a] = 0$

$\Rightarrow$ $a\lambda a = 0$, $\lambda aa = 0$, or $\lambda aaa + aa\lambda a = 0$.

Assume $a\lambda a \neq 0$ and $\lambda aa \neq 0$. Then $\lambda a\lambda aa + a\lambda a\lambda a = 0$ and $\lambda aaa + aa\lambda a = 0$.

Therefore

$$\lambda(\lambda aaa + aa\lambda a) = 0$$

$$\Rightarrow \quad \lambda^2 aaa + \lambda a\lambda aa + \lambda aa\lambda a + \lambda aa\lambda a + a\lambda a\lambda a + aa\lambda^2 a = 0$$

$$\Rightarrow \quad \lambda^2 aaa + 2\lambda aa\lambda a + aa\lambda^2 a = 0$$

$$\Rightarrow \quad a(\lambda^2 aaa + 2\lambda aa\lambda a + aa\lambda^2 a)a = 0.$$

Notice that $\lambda a\lambda aa + a\lambda a\lambda a = 0$ implies $a\lambda^2 aa = 0$ by (6.7).

Thus

$$2a\lambda aa\lambda aa = 0$$

or

$$a\lambda aa\lambda aa = 0.$$

We have shown that either $b\lambda c = 0$, $\forall$ $b,c, \in \delta^3\lambda R$; $\lambda bc = 0$, $\forall$ $b,c \in \delta^3\lambda R$; or $b\lambda cd\lambda ef = 0$, $\forall$ $b,c,d,e,f \in \delta^3\lambda R$. If λ and δ are not nilpotent then $\delta^3\lambda R \neq 0$ by Theorem 4.3. Hence $\lambda\delta^3\lambda R = 0$. But (6.7) implies $\lambda b\lambda yc + b\lambda^2 yc + b\lambda y\lambda c = 0$, $\forall$ $b,c, \in \delta^3\lambda R$ and $\forall$ $y \in R$. It follows that $\lambda^2 y = 0$, $\forall$ $y \in R$, a contradiction. We conclude that either λ or δ must be nilpotent.

7. OPEN QUESTIONS AND REMARKS

<u>Conjecture 7.1</u>: (Chapter 3) Assume R is a prime ring and λ is a derivation of R. We say λ is algebraic over C if $\exists$ a polynomial $p(t) = c_o + c_1 t + \ldots + c_m t^m$, $c_i \in C$, $c_m \neq 0$, such that $p(\lambda)x = (c_o + c_1\lambda + \ldots + c_m\lambda^m)x = 0, \forall\, x \in R$. If λ and λ^n are both derivations of R, then we saw in Section 3.3 that, with appropriate characteristic restrictions, λ is algebraic for $n = 3,4,5$, and 6. An obvious question is whether or not the techniques used in Section 3.3 can be modified or extended for $n \geq 7$. We conjecture that if λ and λ^n, $n > 1$, are both derivations of a prime ring R, with characteristic sufficiently large, then λ is algebraic.

<u>Conjecture 7.2</u>: (Chapter 3) Martindale and Miers [24] have recently proven Conjecture 7.1 to be true if λ and λ^n are inner derivations. As stated in Proposition 3.1, they have not only shown that λ is algebraic, but also the following:

(1) either $\lambda^n = 0$ or the minimal polynomial $\psi(x)$ of λ is semisimple if n is odd.

and (2) $\lambda^{n-1} = 0$ if n is even.

Assume for a moment that the characteristic of R is 0, δ and δ^n are both derivations of R, and Conjecture 7.1 is true for outer as well as inner derivations. Then δ is algebraic. We would also like to conclude, as in Proposition 3.1, that

(1) either $\delta^n = 0$ or the minimal polynomial $\psi(x)$ of δ is semisimple if n is odd.

and (2) $\delta^{n-1} = 0$ if n is even.

Indeed, this is the case if Δ^n is a derivation of Q, where Δ is the unique extension of δ to all of Q. For if Δ^n is a derivation, then by Kharchenko's Proposition 2.2, Δ and Δ^n are both inner derivations. Thus we can apply Proposition 3.1 to Δ and Δ^n, and since Chung, Kovacs, and Luh [4] have recently shown that $\psi_\delta = \psi_\Delta$, we have the desired conclusion. Note that the above depends on Δ^n being a derivation. Therefore we conjecture the following:
if δ and δ^n are both derivations of a prime ring R, then Δ^n is a derivation of Q, where Δ is the unique extension of δ to all of Q.

<u>Conjecture 7.3</u>: (Chapter 4) If λ and δ are derivations of a prime ring R and $\lambda^n\delta^m = 0$ for $n,m \in Z^+$, then either λ or δ is nilpotent.

<u>Conjecture 7.4</u>: (Chapter 5) If λ,δ, and γ are derivations of a prime ring R, characteristic of R is not 2, and $\lambda\delta = \gamma^n$, where n is odd, then either $\lambda = 0$ or $\delta = 0$.

<u>Conjecture 7.5</u>: (Chapter 6) If λ and δ are derivations of a prime ring R and $\lambda\delta^2\lambda = 0$, then either λ or δ is nilpotent.

<u>Conjecture 7.6</u>: (Chapters 3,4,5,and 6) Assume λ_i, δ_j are derivations of a torsion-free, prime ring R and

$$\prod_{i=1}^{t_1} \lambda_i = \prod_{j=1}^{t_2} \delta_j$$ where t_1 is even and t_2 is odd. Then either λ_i is nilpotent for some i or δ_j is nilpotent for some j.

As a final note, assume R is a prime ring and λ is a derivation of R. Let C[t] be the ring of all polynomials in t with coefficients in C, and let Z[y] be the ring of all polynomials in y with

coefficients in Z. We wonder what can be said if $\exists$ polynomials $f \in Z[y]$ and $p \in C[t]$ such that $p(\lambda)x = xf(x)$, $\forall\ x \in R$. Obviously, if $f = 0$, then we simply have the case where λ is algebraic. The following Theorem addresses the special situation where $p(t) = t$.

<u>Theorem 7.7</u> Assume R is a torsion-free ring and $\lambda : R \to R$ is defined by $\lambda(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x$, where $a_i \in Z$, $a_n \neq 0$. If $\lambda(z_1 z_2) = \lambda z_1 z_2 + z_1 \lambda z_2$, $\forall\ z_1, z_2 \in R$, then R is nil.

<u>Proof.</u> If $n = 1$, then $\forall\ x \in R$, $\lambda(x) = a_1 x$

$$\Rightarrow \quad a_1 x^2 = \lambda(x^2) = \lambda xx + x\lambda x = 2a_1 x^2$$

$$\Rightarrow \quad a_1 x^2 = 0$$

$$\Rightarrow \quad x^2 = 0.$$

If $n \geq 2$, then $\forall\ x \in R$, then $\lambda(x^2) = \lambda xx + x\lambda x = 2x\lambda x$

$$\Rightarrow \quad a_n x^{2n} + a_{n-1}x^{2n-2} + \ldots + a_2 x^4 + a_1 x^2$$

$$= 2a_n x^{n+1} + 2a_{n-1}x^n + \ldots + 2a_2 x^3 + 2a_1 x^2$$

$$\Rightarrow \quad a_n x^{2n} = x^2(g(x)) \tag{7.1}$$

where $g(x) = b_k x^k + b_{k-1}x^{k-1} + \ldots + b_1 x + b_o$, $\quad k < 2n - 2$.

Note that if all $b_j = 0$, then $x^{2n} = 0$. Therefore assume at least one $b_j \neq 0$. Multiplying (7.1) by 2^{2n} yields

$$a_n 2^{2n} x^{2n} = 2^{2n} x^2 (b_k x^k + b_{k-1}x^{k-1} + \ldots + b_1 x + b_o).$$

Replacing x by 2x in (7.1) yields

$$a_n 2^{2n} x^{2n} = 2^2 x^2 (b_k 2^k x^k + b_{k-1} 2^{k-1} x^{k-1} + \ldots + b_1 2x + b_o)$$

Subtracting the last equation from the previous one yields

$$(2^{2n} - 2^{k+2}) b_k x^{k+2} + (2^{2n} - 2^{k+1}) b_{k-1} x^{k+1} + \ldots + (2^{2n} - 2^2) b_o x^2 = 0,$$

or we may write $c_k x^{k+2} + c_{k-1} x^{k+1} + \ldots + c_o x^2 = 0$, where not all c_i are zero, say $c_j \neq 0$.

Replacing x by 2x,3x,...,kx, and (k + 1)x we obtain

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2^{k+2} & 2^{k+1} & \cdots & 2^2 \\ 3^{k+2} & 3^{k+1} & \cdots & 3^2 \\ \vdots & \vdots & & \vdots \\ (k+1)^{k+2} & (k+1)^{k+1} & \cdots & (k+1)^2 \end{bmatrix} \begin{bmatrix} c_k x^{k+2} \\ c_{k-1} x^{k+1} \\ c_{k-2} x^k \\ \vdots \\ c_o x^2 \end{bmatrix} = \underline{0} .$$

Since the Vandermonde determinant is not zero in Z, we have $x^{j+2} = 0$.

In [14], it is shown that if R is a ring, $P \neq (0)$ is a right ideal of R, and $\exists\, n \in Z^+$ such that $a^n = 0$, $\forall\, a \in P$, then R contains a nonzero nilpotent ideal. Therefore in Theorem 7.7, if $R \neq 0$ and we consider R itself as a right ideal, then not only is R nil, but R contains a nonzero nilpotent ideal. Such an R cannot be prime. We conclude that the only prime ring satisfying the hypotheses of Theorem 7.7, must be (0).

8. REFERENCES

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